LIMITS OF BC–TYPE ORTHOGONAL POLYNOMIALS AS THE NUMBER OF VARIABLES GOES TO INFINITY

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Abstract. We describe the asymptotic behavior of the multivariate BC–type Jacobi polynomials as the number of variables and the Young diagram indexing the polynomial go to infinity. In particular, our results describe the approximation of the spherical functions of the infinite-dimensional symmetric spaces of type B, C, D or BC by the spherical functions of the corresponding finite–dimensional symmetric spaces. Similar results for the Jack polynomials were established in our earlier paper (Intern. Math. Res. Notices 1998, no. 13, 641–682; arXiv: q-alg/9709011). The main results of the present paper were obtained in 1997.

Contents

1. Introduction

In this paper we evaluate the limit behavior of the (suitably normalized) multivariate orthogonal polynomials associated to the root system BC_n as $n \to \infty$. For given $n = 1, 2, \dots$, these polynomials are viewed as functions on the n–dimensional torus $\mathbb{T}^n$. They are indexed by an arbitrary partition λ of length at most n, and also

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depend on 3 parameters $\theta > 0$, $a > -1$, $b > -1$. The normalization is determined by the condition that the polynomials take value 1 at the point $(1, \dots, 1)$.

We let n go to infinity and assume that the partition $\lambda = \lambda(n)$ varies together with n. We obtain necessary and sufficient conditions on the sequence $\{\lambda(n)\}$ under which the corresponding polynomials uniformly converge on any fixed subtorus $\mathbb{T}^k$, $k = 1, 2, \dots$. We also describe all possible limit functions, which live on an infinite–dimensional torus; it turns out that they depend on countably many continuous parameters.

The motivation for studying this asymptotic problem comes from representation theory of infinite–dimensional classical groups. For certain special values of the parameters θ, a, b (in particular, θ has to be one of the numbers $\frac{1}{2}$, 1, 2) the normalized orthogonal polynomials of type BC_n can be interpreted as the indecomposable spherical functions on rank n symmetric spaces of compact type, with restricted root system B_n, C_n, D_n or BC_n. Likewise, the limit functions can be interpreted as indecomposable spherical functions on certain infinite–dimensional analogs of these symmetric spaces.

Our results not only provide a complete classification of the spherical functions on infinite–dimensional symmetric spaces but also explain how these "infinite–variate" spherical functions are approximated by the conventional ("finite–variate") spherical functions.

The present paper can be viewed as a continuation of our paper [OO4] where we studied a similar asymptotic problem for the Jack polynomials. The Jack polynomials depend on a single continuous parameter $\theta > 0$. For 3 special values $\frac{1}{2}, 1, 2$ of this parameter, the (suitably normalized) Jack polynomials with n variables can be interpreted as indecomposable spherical functions on rank n compact symmetric spaces with restricted root system A_n:

$$U(n)/O(n), \qquad (U(n) \times U(n))/U(n), \qquad U(2n)/Sp(n).$$

In the case $\theta = 1$ the Jack polynomials become the Schur polynomials. The normalized Schur polynomials can also be viewed as the normalized irreducible characters of the groups $U(n)$. The large n asymptotics of these characters was found, for the first time, in the pioneer work of A. M. Vershik and S. V. Kerov, [VK].

The results of our paper [OO4] and of the present paper together provide a far generalization of [VK], involving all families of multivariate orthogonal polynomials connected with classical root systems. A natural field of application for our results is infinite–dimensional noncommutative harmonic analysis in the spirit of the papers Olshanski [O4] and Borodin–Olshanski [BorO]. Those papers present a detailed study of harmonic analysis on the infinite–dimensional unitary group, and they substantially use the large n asymptotics of Schur polynomials. One can expect that a similar theory can be built for other infinite–dimensional classical groups or symmetric spaces, which will necessarily imply a similar use of more general orthogonal polynomials.

We proceed to a more detailed description of the results obtained in the present paper.

1A. BC_n orthogonal polynomials.

Throughout the paper $n = 1, 2, \dots$ denotes a natural number. Let W denote the BC_n Weyl group

$$W = S(n) \ltimes \mathbb{Z}_2^n \,.$$

We shall need two copies W_* and W_+ of this group acting on functions in n variables. The $S(n)$ part in both cases permutes the variables $z_1, \dots, z_n$, and the $\mathbb{Z}_2^n$ part acts by

$$f(z) \mapsto f(z_1^{\pm 1}, \dots, z_n^{\pm 1}), \quad f(z) \mapsto f(\pm z_1, \dots, \pm z_n),$$

in W_* and W_+, respectively.

Given a partition λ, we denote by $\ell(\lambda)$ the number of nonzero parts of λ. The BC_n orthogonal polynomials (also called the BC_n Jacobi polynomials) are certain W_*–invariant Laurent polynomials in n variables, labelled by partitions λ with $\ell(\lambda) \le n$ and depending on 3 parameters

$$\theta > 0, \quad a, b > -1.$$

The polynomials are defined as follows. The parameters θ, a, b specify an inner product,

$$(f, g) = \int_{\mathbb{T}^n} f(z)\, \overline{g(z)}\, \mathfrak{w}(z) \cdot \mathrm{Haar}(dz),$$

of functions on the n-dimensional torus

$$\mathbb{T}^n = \{(z_1, \dots, z_n) \subset \mathbb{C}^n\}, \quad |z_i| = 1,$$

where "Haar" is the Haar measure on $\mathbb{T}^n$ and $\mathfrak{w}(z)$ is the following W_*-invariant weight function

$$\mathfrak{w}(z) = \prod_{1 \le i < j \le n} |z_i - z_j|^{2\theta} |1 - z_i z_j|^{2\theta} \prod_{1 \le i \le n} |1 - z_i|^{2a+1} |1 + z_i|^{2b+1}. \tag{1.1}$$

The polynomials in question, denoted as $\mathfrak{J}_\lambda(z; \theta, a, b)$, are W_*-invariant, orthogonal with respect to the above inner product, and satisfy the triangularity condition

$$\mathfrak{J}_\lambda(z; \theta, a, b) = z^\lambda + \dots, \tag{1.2}$$

where $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ and dots stand for lower monomials in the lexicographic order. These properties characterize the polynomials $\mathfrak{J}_\lambda(z; \theta, a, b)$ uniquely.

Actually, the polynomials $\mathfrak{J}_\lambda(z; \theta, a, b)$ possess a stronger triangularity property. To state it we need some notation. Given a partition μ with $\ell(\mu) \le n$, set

$$\widetilde{m}_\mu(z) = \sum_{\nu \in W_+(\mu)} z^\nu, \qquad z \in \mathbb{T}^n \tag{1.3}$$

(summed over weights ν in the W_+–orbit of μ). This is an analog of the monomial symmetric function for the root system BC_n. Next, let $\varepsilon_1, \dots, \varepsilon_n$ be the canonical basis in $\mathbb{Z}^n$. Write $\mu \ll \lambda$ if the vector $(\lambda_1 - \mu_1, \dots, \lambda_n - \mu_n) \in \mathbb{Z}^n$ can be written as a linear combination of the vectors $\varepsilon_i - \varepsilon_j$ $(1 \le i < j \le n)$ and ε_i $(1 \le i \le n)$ with nonnegative integral coefficients. In this notation, the refinement of the triangularity condition has the form:

$$\mathfrak{J}_\lambda(z_1, \dots, z_n; \theta, a, b) = \sum_{\mu \ll \lambda} u_{\lambda\mu}(\theta, a, b)\, \widetilde{m}_\mu(z_1, \dots, z_n), \tag{1.4}$$

where $u_{\lambda\mu}(\theta, a, b)$ are certain coefficients such that $u_{\lambda\lambda}(\theta, a, b) = 1$.

This is a specialization of the general definition of the multivariate Jacobi polynomials corresponding to an arbitrary root system R. See, for example, Heckman's lectures published in [HS], Macdonald [Ma4], and Koornwinder's expository paper [K2]. Those general polynomials depend on a Weyl group invariant function

$$\alpha \mapsto k_\alpha\,, \quad \alpha \in R\,,$$

on the root system R. In our case

$$R = \{\pm\varepsilon_i \pm \varepsilon_j\,, \pm\varepsilon_i\,, \pm 2\varepsilon_i\}\ ,$$

where $\{\varepsilon_i\}$, as above, is the standard basis of $\mathbb{R}^n$, and

$$k_{\pm\varepsilon_i \pm\varepsilon_j} = 2\theta\,, \quad k_{\pm\varepsilon_i} = a - b\,, \qquad k_{\pm 2\varepsilon_i} = b + \tfrac{1}{2}\,. \tag{1.5}$$

The numbers k_α are viewed as formal root multiplicities. Thus, the "half–sum of the positive roots" is defined as

$$\rho := \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha = (\theta(n-1)+\sigma,\, \theta(n-2)+\sigma,\, \dots,\, \theta+\sigma,\, \sigma)\,,$$

where

$$\sigma = \frac{a+b+1}{2}\,.$$

We shall need the following known fact:

Proposition 1.1. *Assume $a \ge b \ge -\frac{1}{2}$ so that the formal multiplicities defined in* (1.5) *are nonnegative. Then in the expansion* (1.4)*, the coefficients $u_{\lambda\mu}(\theta, a, b)$ are all nonnegative.*

Proof. See Macdonald's paper [Ma4], formula (11.15) and the argument following it. □

By an appropriate specialization of the parameters a, b one can obtain the orthogonal polynomials associated to the root systems B_n, C_n or D_n. Namely, we have to set $b = -\frac{1}{2}$, $a = b \neq -\frac{1}{2}$ or $a = b = -\frac{1}{2}$, respectively. It should be noted, however, that in the D_n case, the polynomial $\mathfrak{J}_\lambda(z; \theta, a, b) = \mathfrak{J}_\lambda(z; \theta, -\frac{1}{2}, -\frac{1}{2})$ with $\lambda_n > 0$ is the sum of certain two "twin" D_n–polynomials.

Note that the orthogonal polynomials for the A_n root system are very closely related to Jack polynomials (see Beerends and Opdam [BeO]); the asymptotics of Jack polynomials as $n \to \infty$ was studied in our paper [OO4]. As for the exceptional root systems, there are, obviously, no $n \to \infty$ asymptotic problems.

Similarly to the Jack polynomials, the polynomials $\mathfrak{J}_\lambda(z; \theta, a, b)$ are eigenfunctions of n commuting differential operators and describe excitations in certain completely integrable quantum many body systems which were introduced by Olshanetsky and Perelomov [OP].

For certain special values of the parameters θ, a, b, the root data $(R, \{k_\alpha\})$ given by (1.5) correspond to the restricted root system of a rank n compact symmetric space $G(n)/K(n)$ of classical type (for more detail, see §6). Then the commuting

differential operators mentioned above become the radial parts of invariant differential operators on $G(n)/K(n)$.

Next, we introduce the *normalized* Jacobi polynomials

$$\Phi_\lambda(z;\theta,a,b) := \frac{\mathcal{J}_\lambda(z_1,\dots,z_n;\theta,a,b)}{\mathcal{J}_\lambda(\underbrace{1,\dots,1}_{n};\theta,a,b)}\,. \tag{1.6}$$

For special values of θ, a, b these are the *indecomposable spherical functions* on the corresponding symmetric space $G(n)/K(n)$.

The Jacobi polynomials $\mathcal{J}_\lambda(z;\theta,a,b)$ are degenerations as $q \to 1$ of the 6–parametric Koornwinder polynomials which are eigenfunctions of certain commuting q-difference operators (see Koornwinder [K1], van Diejen [Di]).

Finally, note that the Jacobi polynomials $\mathcal{J}_\lambda(z;\theta,a,b)$ can be transferred from the torus $\mathbb{T}^n$ to the cube $[-1,1]^n$ via the map

$$z_i \mapsto x_i = \frac{z_i + z_i^{-1}}{2}\,,$$

which takes W_*-invariant polynomials in z to symmetric polynomials in x. The weight function (1.1) is then replaced by the weight function

$$\prod_{1\le i<j\le n} |x_i - x_j|^{2\theta} \prod_{1\le i\le n} (1-x_i)^a (1+x_i)^b$$

with respect to Lebesgue measure dx on the cube. When $n = 1$, this is the familiar weight function for the classical Jacobi polynomials.

1B. Statement of the main result.

Fix some $\theta > 0$. We could have also fixed some $a, b > -1$; however, at no extra cost, we can consider the following more general situation. Namely, we fix two sequences

$$\{a_n\}\,, \{b_n\}\,,$$

such that $a_n, b_n > -1$ and the limits

$$\bar{a} = \lim_{n\to\infty} \frac{a_n}{n}\,, \qquad \bar{b} = \lim_{n\to\infty} \frac{b_n}{n} \tag{1.7}$$

exist. In particular, if a_n and b_n do not depend on n then

$$\bar{a} = \bar{b} = 0.$$

We consider θ, $\{a_n\}$, and $\{b_n\}$ as fixed parameters of our problem and study the limit behavior as $n \to \infty$ of the functions

$$\Phi_{\lambda(n)}(z_1,\dots,z_k,\underbrace{1,\dots,1}_{n-k \text{ times}};\theta,a_n,b_n)\,, \tag{1.8}$$

where $k = 1, 2, 3, \dots$ is fixed and $\{\lambda(n)\}$ is a sequence of partitions with $\ell(\lambda(n)) \le n$,

$$\lambda(n) = (\lambda(n)_1 \ge \lambda(n)_2 \ge \dots \ge \lambda(n)_n \ge 0). \tag{1.9}$$

Definition 1.2. Let $\{\lambda(n)\}$ be a sequence of partitions as in (1.9).

(i) $\{\lambda(n)\}$ is said to be *regular* if for every fixed k, the functions (1.8) uniformly converge on the torus $\mathbb{T}^k$, as $n \to \infty$.

(ii) $\{\lambda(n)\}$ is said to be *infinitesimally regular* if for every fixed k, the Taylor expansions of (1.8) in a local system system of coordinates about the point $(1, \dots, 1) \in \mathbb{T}^k$ have a coefficient–wise limit; in this case we shall say that the functions (1.8) *converge infinitesimally*. In other words, this type of convergence means the convergence of jets at the point $(1, \dots, 1) \in \mathbb{T}^k$.

(iii) $\{\lambda(n)\}$ is said to be *minimally regular* if the functions

$$\Phi_{\lambda(n)}(z, \underbrace{1, \dots, 1}_{n-1 \text{ times}}; \theta, a_n, b_n)\,, \quad |z| = 1\,,$$

converge pointwise to a continuous function on $\mathbb{T}^1$.

Given a partition λ, let λ' denote the conjugate partition. That is, λ'_i is the length of the ith column in the diagram of λ. Let $|\lambda|$ denote the sum of the parts of λ (equivalently, the number of the boxes in the corresponding diagram).

Definition 1.3. A sequence (1.9) is said to be a *Vershik-Kerov sequence* (VK sequence, for short) if the following limits exist:

$$\begin{aligned} \alpha_i &:= \lim \frac{\lambda(n)_i}{n} < \infty\,, \quad i = 1, 2, \dots\,, \\ \beta_i &:= \lim \frac{(\lambda(n))'_i}{n} < \infty\,, \quad i = 1, 2, \dots\,, \\ \delta &:= \lim \frac{|\lambda(n)|}{n} < \infty\,. \end{aligned} \tag{1.10}$$

It is readily checked (see below) that the number

$$\gamma := \delta - \sum (\alpha_i + \beta_i)$$

is nonnegative. The numbers α_i, β_i, γ (or α_i, β_i, δ) are called the *VK parameters* of the sequence $\{\lambda(n)\}$.

Let us give a slightly different (but equivalent) definition of the VK parameters, which makes evident the inequality $\gamma \geq 0$. Let $d(n)$ denote the number of diagonal boxes in the Young diagram corresponding to $\lambda(n)$. For $i = 1, \dots, d(n)$, we replace in (1.10) the row lengths $\lambda(n)_i$ and the column lengths $(\lambda(n))'_i$ by the respective *modified Frobenius coordinates*

$$\lambda(n)_i - i + \tfrac{1}{2}, \qquad (\lambda(n))'_i - i + \tfrac{1}{2},$$

and for $i > d(n)$, we replace the row and column lengths by zeros, which does not affect the definition of α_i and β_i. On the other hand, the sum of the modified Frobenius coordinates equals $|\lambda(n)|$. After the limit transition, this turns into the inequality $\sum(\alpha_i + \beta_i) \leq \delta$, so that $\gamma \geq 0$.

The following is the main result of the present paper.

Theorem 1.4. *Let $\theta > 0$ and $a_n \ge b_n \ge -\frac{1}{2}$ be parameters satisfying* (1.7). *Let $\{\lambda(n)\}_{n=1,2,\dots}$ be a sequence of partitions with $\ell(\lambda(n)) \le n$.*

(i) *All 3 regularity properties of Definition 1.2 are equivalent to each other and are also equivalent to the VK conditions* (1.10) *of Definition 1.3.*

(ii) *If $\{\lambda(n)\}$ is a VK sequence with parameters α_i, β_i, γ then for any fixed $k = 1, 2, \dots$*

$$\lim_{n\to\infty} \Phi_{\lambda(n)}(z_1, \dots, z_k, \underbrace{1, \dots, 1}_{n-k \text{ times}}; \theta, a_n, b_n) = \prod_{j=1}^{k} \phi_{\alpha,\beta,\gamma,\bar a,\bar b}\left(\frac{z_j + z_j^{-1}}{2}\right),$$

where $\bar a, \bar b$ are defined in (1.7), *and $\phi_{\alpha,\beta,\gamma,\bar a,\bar b}$ is the following function of a single variable $x \in [-1, 1]$*

$$\phi_{\alpha,\beta,\gamma,\bar a,\bar b}(x) = e^{\gamma(x-1)} \prod_{i=1}^{\infty} \frac{1 + \dfrac{\beta_i}{2}\left(\dfrac{2\theta + \bar a + \bar b - \theta\beta_i}{\theta + \bar a}\right)(x-1)}{\left(1 - \dfrac{\alpha_i}{2\theta}\left(\dfrac{2\theta + \bar a + \bar b + \alpha_i}{\theta + \bar a}\right)(x-1)\right)^{\theta}}. \tag{1.11}$$

In particular, if $\bar a = \bar b = 0$ (which is the case when a_n and b_n do not depend on n) then the above expression can be written as

$$\phi_{\alpha,\beta,\gamma}\left(\frac{z + z^{-1}}{2}\right) = e^{\frac{\gamma}{2}(z + z^{-1} - 2)} \prod_{i=1}^{\infty} \frac{\left(1 + \frac{\beta_i}{2}(z-1)\right)\left(1 + \frac{\beta_i}{2}(z^{-1}-1)\right)}{\left(\left(1 - \frac{\alpha_i}{2\theta}(z-1)\right)\left(1 - \frac{\alpha_i}{2\theta}(z^{-1}-1)\right)\right)^{\theta}}. \tag{1.12}$$

Note that the infinite products in these formulas are convergent because $\sum(\alpha_i + \beta_i) < \infty$.

Note also that there exists an *a priori* argument (see [O2, §23] and [O3]) explaining why the limit functions in Theorem 1.4 factorize. This argument works for special values of (θ, a, b) when the limit functions admit a representation theoretic interpretation, see §1C below.

The proof of Theorem 1.4 is completed in section 4. The strategy of the proof is the same as in our paper [OO4]. Our main technical tool is the *binomial formula* for the Jacobi polynomials $\mathfrak{J}_\lambda(z; \theta, a, b)$ which involves the so called interpolation BC_n polynomials $I_\mu(x)$. These objects are discussed in detail in section 2 below. Note that the binomial formula we need is a degeneration of a more general binomial formula for the Koornwinder polynomials obtained in [Ok1].

There is, however, a difference in the way we establish the sufficient conditions of regularity in the Jack and Jacobi cases. The argument in the Jack case relies heavily on the use of generating series for one–row shifted Jack polynomials. It is unknown how to evaluate such a series in the Jacobi case. Instead, there is a simple argument available which uses the stability of the polynomials I_μ.

1C. Other results.

In section 5, we fix parameters a, b such that $a \ge b \ge -\frac{1}{2}$. For $n = 1, 2, \dots$ let $\Upsilon_n^{\theta,a,b}$ be the convex set of functions on the torus $\mathbb{T}^n$ which can be written as

convex linear combinations of functions $\Phi_\lambda(z_1,\dots,z_n;\theta,a,b)$ with $\ell(\lambda)\le n$. It turns out that the specialization $z_n=1$ determines an affine map $\Upsilon_n^{\theta,a,b}\to\Upsilon_{n-1}^{\theta,a,b}$. Consequently we can form the projective limit of the convex sets $\Upsilon_n^{\theta,a,b}$ as $n\to\infty$, which we denote as Υ^θ. Consider the set $\mathbb{T}_0^\infty=\varinjlim\mathbb{T}^n$ whose elements are infinite vectors $(z_1,z_2,\dots)\in\mathbb{T}\times\mathbb{T}\times\dots$ with finitely many coordinates z_i distinct from 1. Then elements of Υ^θ can be described as functions $\varphi(z_1,z_2,\dots)$ on $\mathbb{T}_0^\infty$ such that for any $n=1,2,\dots$, the function

$$\varphi_n(z_1,\dots,z_n)=\varphi(z_1,\dots,z_n,1,1,\dots)$$

on $\mathbb{T}^n$ belongs to $\Upsilon_n^{\theta,a,b}$.

It is clear that Υ^θ is a convex set. In Theorem 5.2 we show that there is a one–to–one correspondence between extreme points of Υ^θ and collections (α,β,γ) of VK parameters,

$$\alpha=(\alpha_1\ge\alpha_2\ge\dots\ge0),\quad \beta=(\beta_1\ge\beta_2\ge\dots\ge0),\quad \gamma\ge0,\quad \sum(\alpha_i+\beta_i)<\infty.$$

Given (α,β,γ), the corresponding extremal function on $\mathbb{T}_0^\infty$ has the form

$$\varphi(z_1,z_2,\dots)=\prod_{j=1}^\infty\phi_{\alpha,\beta,\gamma}\left(\frac{z_j+z_j^{-1}}{2}\right),$$

where the functions $\phi_{\alpha,\beta,\gamma}$ are those defined in (1.12). That is, the possible limits of normalized Jacobi polynomials are precisely the extreme points of Υ^θ. In particular, this implies that the set Υ^θ does not depend on parameters a,b.

In section 6, we consider 7 infinite–dimensional symmetric spaces G/K defined as the inductive limits of classical compact symmetric spaces $G(n)/K(n)$ of type B, C, D or BC. We assume that parameters θ,a,b take special values depending on the series $\{G(n)/K(n)\}$. Then the convex set Υ^θ can be identified with the set of positive definite, two–sided K–invariant, normalized functions on G. As a corollary of Theorem 5.2 we obtain an explicit description of the indecomposable spherical functions on G/K. Moreover, Theorem 1.4 shows how these functions are approximated by the indecomposable spherical functions of the rank n compact symmetric spaces $G(n)/K(n)$.

An important particular case of this result, corresponding to the large n asymptotic behavior of the characters of the orthogonal and symplectic groups, was earlier obtained by Boyer [Boy].

In section 7 we present an elementary derivation of two basic facts about the Jacobi polynomials $\mathcal{J}_\lambda$, the binomial formula and the branching rule, for the case $\theta=1$.

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2. Interpolation BC_n polynomials and binomial formula

2A. Interpolation BC_n polynomials.

Fix $\theta>0$ and denote by Λ_n^θ the algebra of polynomials in n variables $x_1,\dots,x_n$, symmetric in variables $x_i-\theta i$, $i=1,\dots,n$. Consider the projective limit of these algebras

$$\Lambda^\theta=\varprojlim\Lambda_n^\theta$$

taken in the category of filtered algebras, with respect to homomorphisms sending the last variable to 0. Here the filtration is defined by the total degree $\deg(\,\cdot\,)$ of a polynomial. That is, an element $f \in \Lambda^\theta$ is a sequence of polynomials $f_n \in \Lambda_n^\theta$ such that $f_{n+1}(x_1, \dots, x_n, 0) = f_n(x_1, \dots, x_n)$ and $\deg(f_n)$ remains bounded. Elements $f \in \Lambda^\theta$ can be evaluated at any infinite vector $x = (x_1, x_2, \dots)$ with finitely many nonzero coordinates. In particular, for any partition λ, the value $f(\lambda)$ is well defined. Let h be one more variable and consider the algebras

$$\Lambda^\theta(h) = \Lambda^\theta \otimes \mathbb{C}(h)\,, \quad \Lambda^\theta[h] = \Lambda^\theta \otimes \mathbb{C}[h]\,,$$

and define $\Lambda_n^\theta(h)$ and $\Lambda_n^\theta[h]$ similarly.

Proposition 2.1. (i) *In the $\mathbb{C}(h)$–algebra $\Lambda_n^\theta(h)$ defined above, there exist polynomials $I_\mu(x_1, \dots, x_n; \theta; h)$ indexed by arbitrary partitions μ with $\ell(\mu) \le n$, satisfying the following Newton interpolation conditions:*

(1) $\deg(I_\mu(\,\cdot\,; \theta; h)) = 2|\mu|$,
(2) $I_\mu(x_1, \dots, x_n; \theta; h)$ *is W_+–invariant in variables $x_i - \theta i + h$, $i = 1, \dots, n$,*
(3) $I_\mu(\lambda; \theta; h) = 0$ *if $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition with $\ell(\lambda) \le n$, such that $\mu \not\subseteq \lambda$,*
(4) $I_\mu(\mu; \theta; h) \ne 0$.

(ii) *These polynomials are unique up to scalar factors.*

Here the notation $\mu \not\subseteq \lambda$ means that the diagram of μ is not contained in the diagram of λ.

Proof. Any polynomial in $x_1, \dots, x_n$ which is W_+–invariant in variables $x_i - \theta i + h$, $i = 1, \dots, n$, can be viewed simply as a symmetric polynomial in new variables $(x_i - \theta i + h)^2$. Then the claims of the proposition become a particular case of the results of [Ok2], corresponding to the case of the "perfect grid of class II". □

Note that the existence of the polynomials $I_\mu(x_1, \dots, x_n; \theta; h)$ can be derived from the results of Okounkov [Ok1] (see also Rains' paper [R] which contains a different approach to the results of [Ok1]). Namely,

$$I_\mu(x_1, \dots, x_n; \theta; h) = \lim_{q \to 1} \frac{P_\mu^*(q^{x_1}, \dots, q^{x_n}; q, q^\theta, q^{h - n\theta})}{(q-1)^{2|\mu|}}\,, \tag{2.1}$$

where the polynomial in the numerator is the BC_n type shifted (or interpolation) Macdonald polynomial defined in [Ok1, Definition 1.3].

The polynomials $I_\mu(x_1, \dots, x_n; \theta; h)$ are normalized by setting

$$I_\mu(\mu; \theta; h) = \prod_{(i,j) \in \mu} \big(1 + \mu_i - j + \theta(\mu'_j - i)\big)\big(2h - 1 + \mu_i + j - \theta(\mu'_j + i)\big)\,, \tag{2.2}$$

where the product is over all squares (i, j) in the diagram of μ. A motivation for such a normalization is given in [Ok2, Proposition 2.9]. Note that this normalization

is well adapted to the combinatorial formula (2.4). We shall call the polynomials $I_\mu(x_1,\dots,x_n;\theta;h)$ the *interpolation BC_n polynomials*.

The next result provides a *combinatorial formula* for the interpolation BC_n polynomials. It is convenient to state it in terms of *reverse* tableaux (cf. [OO2]). A reverse tableau T of shape μ with entries in $\{1,\dots,n\}$ is defined as a function $T(i,j)$ assigning to each box $(i,j)\in\mu$ a number $T(i,j)\in\{1,\dots,n\}$, such that the numbers decrease strictly down each column and weakly from left to right along each row. The only difference with the conventional definition of a semistandard tableau consists in inverting the natural order in the set of indices $1,\dots,n$.

First, recall the combinatorial formula for the Jack polynomials with parameter θ:

$$P(x_1,\dots,x_n;\theta)=\sum_T \psi_T(\theta)\prod_{(i,j)\in\mu} x_{T(i,j)}\,, \tag{2.3}$$

summed over all reverse tableaux T of shape μ, as defined above, where $\psi_T(\theta)$ is a certain weight factor, which is a rational function in θ (see formulas (7.13'), (10.10), (10.11), and (10.12) in chapter VI of Macdonald's book [Ma3], and also [OO2]; note that our parameter θ is inverse to the parameter α used in [Ma3]). We do not need the explicit expression for $\psi_T(\theta)$.

Proposition 2.2. *We have*

$$\begin{aligned} I_\mu(x_1,\dots,x_n;\theta;h)=\sum_T \psi_T(\theta)\prod_{(i,j)\in\mu} \Big[\big(x_{T(i,j)}+h-\theta T(i,j)\big)^2 \\ -\big((j-1)-\theta(i-1)+h-\theta T(i,j)\big)^2\Big]\,, \end{aligned} \tag{2.4}$$

summed over all reverse tableaux T of shape μ, with entries in $\{1,\dots,n\}$, where $\psi_T(\theta)$ is the same weight factor as in (2.3).

Proof. Using (2.1), this can be obtained as a degeneration of the combinatorial formula for the BC_n interpolation Macdonald polynomials, established in Okounkov [Ok1, Theorem 5.2]. See also [Ok2] and Rains [R]. □

Note that the expression for $I_\mu(\mu;\theta;h)$ given in (2.2) can be obtained from the combinatorial formula (2.4). Indeed, if $(x_1,\dots,x_n)=(\mu_1,\dots,\mu_n)$ then all products in the right–hand side, except a single one (corresponding to a special choice of T) vanish, and the only nonvanishing term gives the expression (2.2). This can be shown using the same argument as in the second proof of Theorem 11.1 from Okounkov–Olshanski [OO1].

We list a number of corollaries of these two propositions.

From the combinatorial formula (2.4) it follows immediately that the polynomials $I_\mu(\,\cdot\,;\theta;h)$, which are initially defined as elements of $\Lambda_n^\theta(h)$, actually belong to $\Lambda_n^\theta[h]$.

Next, the characterization of these polynomials given in Proposition 2.1 shows that they are stable:

$$I_\mu(x_1,\dots,x_n,0;\theta;h)=I_\mu(x_1,\dots,x_n;\theta;h),$$

and, therefore, the sequence $\{I_\mu(x_1,\dots,x_n;\theta;h)\}_{n\ge\ell(\mu)}$ correctly defines an element of the algebra $\Lambda^\theta[h]$. This element will be denoted as $I_\mu(x_1,x_2,\dots;\theta;h)$ or $I_\mu(\,\cdot\,;\theta;h)$ or else simply I_μ.

Let us extend the filtration from Λ^θ to $\Lambda^\theta[h]$ by setting $\deg h=1$. It is clear that the associated graded algebra $\operatorname{gr}\Lambda^\theta[h]$ is naturally isomorphic to the algebra $\Lambda[h]$, where Λ denotes the algebra of symmetric functions in infinitely many variables.

Given an element $f\in\Lambda^\theta[h]$, let us denote by $[f]$ its highest degree term, which is a homogeneous element of the algebra $\Lambda[h]$. We remark that the element $I_\mu\in\Lambda^\theta[h]$ has degree $2|\mu|$ and it follows from the comparison of (2.3) and (2.4) that

$$[I_\mu(x_1,x_2,\dots;\theta;h)]=P_\mu\big(x_1(x_1+2h),\,x_2(x_2+2h),\dots;\theta\big), \tag{2.5}$$

where $P_\mu(\,\cdot\,;\theta)\in\Lambda$ is the Jack symmetric function. Note that for any homogeneous symmetric function $g(x_1,x_2,\dots)$ of degree m, the expression

$$g(x_1(x_1+2h),\,x_2(x_2+2h),\dots)$$

is a well defined homogeneous element of the algebra $\Lambda[h]$ of degree $2m$.

It is also clear that

$$I_\mu(\mu;\theta;h)=H(\mu;\theta)(2h)^{|\mu|}+\dots \tag{2.6}$$

where the dots stand for the lower degree terms in h and $H(\mu;\theta)$ is the following hook–length product

$$H(\mu;\theta):=\prod_{(i,j)\in\mu}((\mu_i-j)-\theta(\mu'_j-i)+1)\,.$$

2B. Binomial formula.

Here by the binomial formula we mean an expansion of the normalized Jacobi polynomials $\Phi_\lambda(z_1,\dots,z_n;\theta,a,b)$ (see (1.6)) about the point $(1,\dots,1)$.

Proposition 2.3. *Let $\ell(\lambda)\le n$. We have*

$$\Phi_\lambda(z_1,\dots,z_n;\theta,a,b)=\sum_\mu\frac{I_\mu(\lambda;\theta;\sigma+\theta n)\,P_\mu(z_1+z_1^{-1}-2,\dots,z_n+z_n^{-1}-2;\theta)}{C(n,\mu;\theta;a,b)}\,. \tag{2.7}$$

where $I_\mu(\lambda;\theta;\sigma+\theta n)$ is the result of specializing $h=\sigma+\theta n$ in $I_\mu(\lambda;\theta;h)$, $\sigma=(a+b+1)/2$, and

$$C(n,\mu;\theta;a,b)=I_\mu(\mu;\theta;\sigma+\theta n)\,\mathcal{J}_\mu(\underbrace{1\dots,1}_{n};\theta,a,b). \tag{2.8}$$

By virtue of condition 3 in Proposition 2.1, the summation in (2.7) actually goes on the finite set of μ's such that $\mu\subseteq\lambda$. An explicit expression for $C(n,\mu;\theta;a,b)$ is given in Remark 2.5 below.

Proof. The expansion (2.7) is a limit case of the binomial formula for Koornwinder polynomials obtained in [Ok1, Theorem 7.1]. Namely, the Koornwinder polynomials are orthogonal on $\mathbb{T}^n$ with weight

$$\prod_{i<j\le n}\frac{(z_i^{\pm1}z_j^{\pm1})_\infty}{(tz_i^{\pm1}z_j^{\pm1})_\infty}\prod_{i=1}^{n}\frac{(z_i^{\pm1},-z_i^{\pm1},q^{1/2}z_i^{\pm1},-q^{1/2}z_i^{\pm1})_\infty}{(a_1z_i^{\pm1},-a_2z_i^{\pm1},q^{1/2}a_3z_i^{\pm1},-q^{1/2}a_4z_i^{\pm1})_\infty},$$

where $q,t,a_1,\dots,a_4$ are the 6 parameters and, by definition,

$$(u_1,u_2,\dots)_\infty=\prod_k\prod_{i=0}^{\infty}(1-q^iu_k)\,.$$

If one sets

$$t=q^\theta\,,\quad a_1=q^{a+\frac12}\,,\quad a_2=q^{b+\frac12}\,,\quad a_3=a_4=0\,,$$

and lets $q\to1$ then these 6-parametric polynomials become $\mathcal{J}_\lambda(z;\theta,a,b)$. Using this, (2.1), and the following immediate corollary of Theorem 5.2 in [Ok1]

$$\lim_{q\to1}P^*_\mu(z;q,q^\theta,q^{(a+b+1)/2})=P_\mu(z_1+z_1^{-1}-2,\dots,z_n+z_n^{-1}-2;\theta)$$

one obtains (2.7) from Theorem 7.1 in [Ok1]. Here P^*_μ stands for the 3-parametric BC_n-type interpolation Macdonald polynomial defined in [Ok1, Definition 1.3]; it has already appeared in (2.1).

Alternatively, one can prove (2.7) without going into q-analogs by just repeating the proof of main theorem of [OO2] and using the fact that the algebra of commuting differential operators whose eigenfunctions are $\mathcal{J}_\lambda$ is isomorphic under the Harish–Chandra homomorphism to the algebra of polynomials in $\lambda_1,\dots,\lambda_n$ that are W_+-invariant in variables

$$\lambda+\rho=(\lambda_1+(n-1)\theta+\sigma,\dots,\lambda_{n-1}+\theta+\sigma,\lambda_n+\sigma)\,.$$

□

Note that the expansion of Jacobi polynomials in Jack polynomials was obtained much earlier by James and Constantine [JK] for the case when $\theta=\frac12$ or 1. However,

in their formula the coefficients of the expansion are written in quite a different form, as certain combinatorial sums over standard tableaux of shape λ/μ (this is *not* equivalent to combinatorial formula (2.4)). Their result was extended to arbitrary values of θ by Macdonald (unpublished work [Ma1, §9]; see also [BeO, §5]) and Lassalle [L]. A discussion of the role of binomial formulas for the characters of the orthogonal and symplectic groups can be found in [OO3].

2C. Asymptotics of denominators in binomial formula.

We fix an arbitrary partition μ and let n go to infinity. As in Theorem 1.4, we assume that the parameters a, b may depend on n. We write them as a_n, b_n and assume that the limits (1.7) exist.

Proposition 2.4. *The denominator* (2.8) *in* (2.7) *has the following asymptotics*

$$C(n,\mu;\theta;a_n,b_n) \sim \frac{H(\mu;\theta)}{H'(\mu;\theta)}\, 4^{|\mu|}\,\theta^{|\mu|}\,(\theta+\bar a)^{|\mu|}\cdot n^{2|\mu|}\,, \tag{2.9}$$

where

$$H(\mu;\theta)=\prod_{(i,j)\in\mu}((\mu_i-j)-\theta(\mu'_j-i)+1), \qquad H'(\mu;\theta)=\prod_{(i,j)\in\mu}((\mu_i-j)-\theta(\mu'_j-i)+\theta)\,,$$

and $\bar a = \lim a_n/n$ *as in* (1.7).

Proof. Recall that $C(n,\mu;\theta;a_n,b_n)$ is the product of two terms, $I_\mu(\mu;\theta;\sigma_n+\theta n)$ and $\mathfrak{J}_\mu(1,\dots,1;\theta,a_n,b_n)$, where $\sigma_n=(a_n+b_n+1)/2$. We claim that, as $n\to\infty$, the following two asymptotic relations hold

$$I_\mu(\mu;\theta;\sigma_n+\theta n)\ \sim\ H(\mu;\theta)\,2^{|\mu|}\,(\theta+\bar\sigma)^{|\mu|}n^{|\mu|}\,, \tag{2.10}$$

$$\mathfrak{J}_\mu(\underbrace{1,\dots,1}_{n};\theta,a_n,b_n)\ \sim\ \frac{1}{H'(\mu;\theta)}\,2^{|\mu|}\left(\frac{\theta+\bar a}{\theta+\bar\sigma}\right)^{|\mu|}\theta^{|\mu|}\,n^{|\mu|}\,, \tag{2.11}$$

where

$$\bar\sigma=\lim_{n\to\infty}\frac{\sigma_n}{n}=\frac{\bar a+\bar b}{2}\,.$$

Clearly, (2.10) and (2.11) imply (2.9).

The first relation immediately follows from (2.6), let us check the second relation.

The following is the general formula, due to Opdam, for the value of a multivariate Jacobi polynomial, indexed by a weight μ, at the unit element, see [HS], Part I, Theorem 3.6.6,

$$\prod_{\alpha>0}\frac{\Gamma\left((\mu+\rho,\alpha^\vee)+k_\alpha+\frac12 k_{\alpha/2}\right)}{\Gamma\left((\mu+\rho,\alpha^\vee)+\frac12 k_{\alpha/2}\right)}\frac{\Gamma\left((\rho,\alpha^\vee)+\frac12 k_{\alpha/2}\right)}{\Gamma\left((\rho,\alpha^\vee)+k_\alpha+\frac12 k_{\alpha/2}\right)}\,,$$

where $\alpha^\vee$ stands for the root dual to α, and $k_{\alpha/2}=0$ if the root $\alpha/2$ does not exist.

In our case, the polynomial in question is just $\mathcal{J}_\mu(\,\cdot\,;\theta,a_n,b_n)$, and the unit element is identified with the point $(1,\dots,1)$. Next, we have

$$\rho=((n-1)\theta+\sigma_n,\dots,\theta+\sigma_n,\sigma_n)$$

and there are 4 types of the positive roots α

$$\varepsilon_i-\varepsilon_j\,,\quad \varepsilon_i+\varepsilon_j\quad (1\le i<j\le n),\qquad \varepsilon_i\,,\quad 2\varepsilon_i\quad (1\le i\le n)$$

with formal multiplicities

$$k_{\varepsilon_i\pm\varepsilon_j}=2\theta,\quad k_{\varepsilon_i}=a_n-b_n\,,\quad k_{2\varepsilon_i}=b_n+\tfrac12\,.$$

As the scalar product used in Opdam's formula we may take the natural scalar product in $\mathbb{R}^n$. Then the dual roots $\alpha^\vee$ are as follows

$$(\varepsilon_i\pm\varepsilon_j)^\vee=\varepsilon_i\pm\varepsilon_j\,,\quad \varepsilon_i^\vee=2\varepsilon_i\,,\quad (2\varepsilon_i)^\vee=\varepsilon_i\,.$$

We split the product over $\alpha>0$ into 4 products according to these 4 types of positive roots

$$\mathcal{J}_\mu(\underbrace{1,\dots,1}_{n\text{ times}};\theta,a_n,b_n)=\prod\nolimits^{(+-)}\prod\nolimits^{(++)}\prod\nolimits^{(+)}\prod\nolimits^{(+2)}\,, \tag{2.12}$$

where

$$\prod\nolimits^{(+-)}=\prod_{1\le i<j\le n}\frac{\Gamma(\mu_i-\mu_j+\theta(j-i+1))}{\Gamma(\mu_i-\mu_j+\theta(j-i))}\frac{\Gamma(\theta(j-i))}{\Gamma\theta(j-i+1))}\,,$$

$$\prod\nolimits^{(++)}=\prod_{1\le i<j\le n}\frac{\Gamma(\mu_i+\mu_j+\theta(2n-i-j+1)+2\sigma_n)}{\Gamma(\mu_i+\mu_j+\theta(2n-i-j)+2\sigma_n)}\frac{\Gamma(\theta(2n-i-j)+2\sigma_n)}{\Gamma(\theta(2n-i-j+1)+2\sigma_n)}\,,$$

$$\prod\nolimits^{(+)}=\prod_{1\le i\le n}\frac{\Gamma(2\mu_i+2\theta(n-i)+2a_n+1)}{\Gamma(2\mu_i+2\theta(n-i)+2\sigma_n)}\frac{\Gamma(2\theta(n-i)+2\sigma_n)}{\Gamma(2\theta(n-i)+2a_n+1)}\,,$$

$$\prod\nolimits^{(+2)}=\prod_{1\le i\le n}\frac{\Gamma(\mu_i+\theta(n-i)+2\sigma_n)}{\Gamma(\mu_i+\theta(n-i)+a_n+\frac12)}\frac{\Gamma(\theta(n-i)+a_n+\frac12)}{\Gamma(\theta(n-i)+2\sigma_n)}\,.$$

It is clear that $\prod^{(+-)}$ is just the formula for the value of the Jack polynomial (which is essentially the Jacobi polynomial for the A series). Therefore (see Stanley [St] or Macdonald [Ma3], VI.10.12)

$$\prod\nolimits^{(+-)}\sim H'(\mu;\theta)^{-1}\theta^{|\mu|}\,n^{|\mu|}\,.$$

Next, we have

$$\prod\nolimits^{(+)}\sim\left(\frac{\theta n+a_n}{\theta n+\sigma_n}\right)^{2|\mu|}\sim\left(\frac{\theta+\bar a}{\theta+\bar\sigma}\right)^{2|\mu|},$$

$$\prod\nolimits^{(+2)}\sim\left(\frac{\theta n+2\sigma_n}{\theta n+a_n}\right)^{|\mu|}\sim\left(\frac{\theta+2\bar\sigma}{\theta+\bar a}\right)^{|\mu|}.$$

Finally, consider the product $\prod^{(++)}$. All factors with $\ell(\mu) < i < j$ are trivial (equal to 1). Moreover, the contribution of every fixed pair $i < j$ is asymptotically trivial, so that the whole contribution of the factors with $1 \le i < j \le \ell(\mu)$ is trivial. Therefore,

$$\prod^{(++)} \sim \prod_{\substack{1\le i\le \ell(\mu)\\ \ell(\mu)<j\le n}} \frac{\Gamma(\mu_i+\theta(2n-i-j+1)+2\sigma_n)}{\Gamma(\mu_i+\theta(2n-i-j)+2\sigma_n)} \frac{\Gamma(\theta(2n-i-j)+2\sigma_n)}{\Gamma(\theta(2n-i-j+1)+2\sigma_n)} .$$

For every i the product over j telescopes to

$$\begin{aligned} &\frac{\Gamma(\mu_i+\theta(2n-i-\ell(\mu))+2\sigma_n)}{\Gamma(\mu_i+\theta(n-i)+2\sigma_n)} \frac{\Gamma(\theta(n-i)+2\sigma_n)}{\Gamma(\mu_i+\theta(2n-i-\ell(\mu))+2\sigma_n)} \\ &\qquad \sim \left(\frac{2\theta n+2\sigma_n}{\theta n+2\sigma_n}\right)^{\mu_i} \sim \left(\frac{2\theta+2\bar\sigma}{\theta+2\bar\sigma}\right)^{\mu_i} , \end{aligned}$$

so that

$$\prod^{(++)} \sim 2^{|\mu|} \left(\frac{\theta+\bar\sigma}{\theta+2\bar\sigma}\right)^{|\mu|} .$$

It follows that

$$\prod^{(++)}\prod^{(+)}\prod^{(+2)} \sim 2^{|\mu|} \left(\frac{\theta+\bar a}{\theta+\bar\sigma}\right)^{|\mu|} .$$

This gives formula (2.11) and concludes the proof. $\square$

Remark 2.5. Using (2.12) one can derive the following explicit expression for the quantity (2.8)

$$C(n,\mu;\theta;a,b) = 4^{|\mu|} \frac{H(\mu;\theta)}{H'(\mu;\theta)} \prod_{i=1}^{n} \frac{\Gamma(\mu_i+(n-i+1)\theta)\Gamma(\mu_i+(n-i)\theta+a+1)}{\Gamma((n-i+1)\theta)\Gamma((n-i)\theta+a+1)} .$$

In particular, $C(n,\mu;\theta;a,b)$ actually does not depend on b. The asymptotic relation (2.9) is readily obtained from this expression.

3. Sufficient conditions of regularity

Recall that we have fixed some $\theta > 0$. Define elements $g_1, g_2, \dots$ in the algebra Λ of symmetric functions by means of a generating function

$$1 + \sum_{k=1}^{\infty} g_k(x_1,x_2,\dots)t^k = \prod_{j=1}^{\infty}(1-x_j t)^{-\theta} ,$$

where t is a formal variable. These elements are algebraically independent generators of Λ. Given some VK parameters α, β, γ, define an algebra homomorphism

$$\epsilon_{\alpha,\beta,\gamma} : \Lambda \longrightarrow \mathbb{C}$$

by

$$1+\sum_{k=1}^{\infty} g_k t^k \xrightarrow{\epsilon_{\alpha,\beta,\gamma}} e^{\gamma\theta t}\prod_{i=1}^{\infty}\frac{1+\beta_i\theta t}{(1-\alpha_i t)^{\theta}}$$

(cf. the definition of the extended symmetric functions, see [OO4, Section 2.7] or [KOO]). In a less formal way, this homomorphism can be written as

$$\prod_{j=1}^{\infty}(1-x_j t)^{-\theta} \xrightarrow{\epsilon_{\alpha,\beta,\gamma}} e^{\gamma\theta t}\prod_{i=1}^{\infty}\frac{1+\beta_i\theta t}{(1-\alpha_i t)^{\theta}}\,.$$

We will also use such a notation in the sequel.

Proposition 3.1. *Let $\{\lambda(n)\}$ be a VK sequence of partitions with parameters α,β,γ, see Definition 1.3, and let $\{h_n\}$ be a sequence of complex numbers such that the limit*

$$\bar h = \lim_{n\to\infty}\frac{h_n}{n}$$

exists. Then for any element $f\in\Lambda^\theta[h]$ we have

$$\lim_{n\to\infty}\frac{f(\lambda(n);h_n)}{n^{\deg f}} = \epsilon_{\alpha,\beta,\gamma}\left(\big[f\big]\Big|_{h=\bar h}\right)\,,$$

where $\big[f\big]\in\Lambda[h]$ is the highest degree term of f.

Proof. Assume first that f does not depend on h, that is, $f\in\Lambda^\theta$. Then $\big[f\big]\in\Lambda$, and the claim is that

$$\lim_{n\to\infty}\frac{f(\lambda(n))}{n^{\deg f}} = \epsilon_{\alpha,\beta,\gamma}\left(\big[f\big]\right)\,,$$

which follows from Theorem 7.1 in [KOO] (see also Theorem 3.1 in [OO4]). Returning to the general case, we write $f\in\Lambda^\theta[h]$ as a polynomial in h,

$$f = f_0 + f_1 h + f_2 h^2 + \cdots + f_m h^m, \qquad f_i\in\Lambda^\theta,$$

and apply the above formula to each $f_0,\dots,f_m$. $\square$

Given $\tau\in\mathbb{C}$, define a homomorphism

$$\pi_\tau:\Lambda\to\Lambda$$

by the formula

$$(\pi_\tau f)(x_1,x_2,\dots)\mapsto f(x_1(x_1+\tau),x_2(x_2+\tau),\dots)\,,\qquad f\in\Lambda\,. \tag{3.1}$$

Then we have the following

Proposition 3.2.

$$\epsilon_{\alpha,\beta,\gamma}\,\pi_\tau\left(\prod_{j=1}^{\infty}\frac{1}{(1-x_jt)^\theta}\right)=e^{\gamma\theta\tau t}\prod_{i=1}^{\infty}\frac{1+\theta\beta_i(\tau-\theta\beta_i)t}{(1-\alpha_i(\tau+\alpha_i)t)^\theta}\,.$$

Proof. By the definition of π_τ and $\epsilon_{\alpha,\beta,\gamma}$ we have

$$\begin{aligned}\epsilon_{\alpha,\beta,\gamma}\pi_\tau\left(\prod_{i=1}^{\infty}\frac{1}{(1-x_jt)^\theta}\right)&=\epsilon_{\alpha,\beta,\gamma}\left(\prod_{j=1}^{\infty}\frac{1}{(1-x_j(x_j+\tau)t)^\theta}\right)\\&=\epsilon_{\alpha,\beta,\gamma}\left(\prod_{j=1}^{\infty}\frac{1}{((1-x_jt_1)(1-x_jt_2))^\theta}\right),\end{aligned}$$

where $t_1+t_2=\tau t$ and $t_1t_2=-t$,

$$\begin{aligned}&=e^{\gamma\theta(t_1+t_2)}\prod_{i=1}^{\infty}\frac{1+\beta_i\theta t_1}{(1-\alpha_it_1)^\theta}\,\frac{1+\beta_i\theta t_2}{(1-\alpha_it_2)^\theta}\\&=e^{\gamma\theta\tau t}\prod_{i=1}^{\infty}\frac{1+\theta\beta_i(\tau-\theta\beta_i)t}{(1-\alpha_i(\tau+\alpha_i)t)^\theta}\,,\end{aligned}$$

which is the result stated. □

Now we can prove the following

Theorem 3.3. *Assume that the condition* (1.7) *is fulfilled. Let* $\{\lambda(n)\}$ *be a Vershik–Kerov sequence with parameters* α,β,γ*, see Definition 1.3. For any fixed* $k=1,2,\dots,$ *the functions*

$$\Phi_{\lambda(n)}(z_1,\dots,z_k,\underbrace{1,\dots,1}_{n-k\text{ times}};\theta,a_n,b_n) \tag{3.2}$$

converge infinitesimally, in the sense of Definition 1.2, to the function

$$\prod_{l=1}^{k}\phi_{\alpha,\beta,\gamma,\bar a,\bar b}\left(\frac{z_l+z_l^{-1}}{2}\right), \tag{3.3}$$

where $\bar a=\lim a_n/n$, $\bar b=\lim b_n/n$ *as in* (1.7)*, and* $\phi_{\alpha,\beta,\gamma,\bar a,\bar b}$ *is the following function of a single variable* $x\in[-1,1]$

$$\phi_{\alpha,\beta,\gamma,\bar a,\bar b}(x)=e^{\gamma(x-1)}\prod_{i=1}^{\infty}\frac{1+\dfrac{\beta_i}{2}\left(\dfrac{2\theta+\bar a+\bar b-\theta\beta_i}{\theta+\bar a}\right)(x-1)}{\left(1-\dfrac{\alpha_i}{2\theta}\left(\dfrac{2\theta+\bar a+\bar b+\alpha_i}{\theta+\bar a}\right)(x-1)\right)^\theta}\,.$$

Proof. Apply to the functions (3.2) the binomial formula (2.7). In that formula μ ranges over partitions with $\ell(\,\cdot\,) \le n$. However, in our case, the last $n-k$ arguments in $P_\mu(\,\cdot\,;\theta)$ equal 0. By the stability of Jack polynomials, this implies that the summation actually goes over partitions μ with $\ell(\mu) \le k$. Thus, we obtain

$$\Phi_{\lambda(n)}(z_1,\dots,z_k,\underbrace{1,\dots,1}_{n-k\text{ times}};\theta,a_n,b_n) = \\ = \sum_{\mu,\ell(\mu)\le k} \frac{I_\mu(\lambda(n);\theta;\sigma_n+\theta n)\,2^{|\mu|}}{C(n,\mu;\theta,a_n,b_n)}\,P_\mu(x_1-1,\dots,x_k-1;\theta) \quad (3.4)$$

where

$$x_l = \frac{z_l + z_l^{-1}}{2}\,, \quad l = 1,\dots,k\,.$$

Since the polynomials $P_\mu(\,\cdot\,;\theta)$ form a homogeneous basis in the space of symmetric polynomials, the infinitesimal convergence of the left–hand side of (3.4) in the sense of Definition 1.2, as $n \to \infty$, is equivalent to the coefficient–wise convergence of the expansion in the right–hand side. Thus, we have to examine the asymptotics of the quantities

$$\frac{I_\mu(\lambda(n);\theta;\sigma_n+\theta n)\,2^{|\mu|}}{C(n,\mu;\theta;a_n,b_n)}, \qquad \mu \text{ fixed}, \ n\to\infty.$$

By Proposition 2.4,

$$C(n,\mu;\theta;a_n,b_n) \sim \frac{H(\mu;\theta)}{H'(\mu;\theta)}\,4^{|\mu|}\,\theta^{|\mu|}\,(\theta+\bar a)^{|\mu|}\cdot n^{2|\mu|}\,,$$

Next, by Proposition 3.1 we have

$$\lim_{n\to\infty} \frac{I_\mu(\lambda(n);\sigma_n+\theta n)}{n^{2|\mu|}} = \epsilon_{\alpha,\beta,\gamma}\left(\big[I_\mu(\,\cdot\,;h)\big]\Big|_{h=\theta+\bar\sigma}\right)\,,$$

and by (2.5) and (3.1)

$$\big[I_\mu(\,\cdot\,;h)\big]\Big|_{h=\theta+\bar\sigma} = \pi_{2(\theta+\bar\sigma)} P_\mu(\,\cdot\,;\theta),$$

which implies

$$\lim_{n\to\infty} \frac{I_\mu(\lambda(n);\sigma_n+\theta n)}{n^{2|\mu|}} = \epsilon_{\alpha,\beta,\gamma}\,\pi_{2\theta+2\bar\sigma}\,P_\mu(\,\cdot\,;\theta).$$

Therefore, the expansion (3.4) converges coefficient–wise to

$$\varepsilon_{\alpha,\beta,\gamma}\pi_{2(\theta+\bar\sigma)}\left(\sum_\mu \frac{H'(\mu)}{H(\mu)}\,P_\mu(y_1,y_2,\dots;\theta)\,P_\mu\left(\frac{x_1-1}{2\theta(\theta+\bar a)},\dots,\frac{x_k-1}{2\theta(\theta+\bar a)};\theta\right)\right)\,, \quad (3.5)$$

where the homomorphisms $\varepsilon_{\alpha,\beta,\gamma}$ and $\pi_{2(\theta+\bar\sigma)}$ act on the variables $y_1, y_2, \dots$. The sum in the round brackets equals

$$\sum_\mu Q_\mu(y_1, y_2, \dots; \theta)\, P_\mu\left(\frac{x_1-1}{2\theta(\theta+\bar a)}, \dots, \frac{x_k-1}{2\theta(\theta+\bar a)}; \theta\right) = \prod_{l=1}^k \prod_{j=1}^\infty \left(1 - \frac{y_j(x_l-1)}{\theta(\theta+\bar a)}\right)^{-\theta}.$$

Here we have used the Cauchy identity for Jack polynomials (see Macdonald [Ma3, ch. VI, (4.13) and §10]). Therefore (3.5) equals

$$\prod_{l=1}^k \varepsilon_{\alpha,\beta,\gamma}\pi_{2\theta+2\bar\sigma}\left(\prod_{j=1}^\infty \left(1 - \frac{y_j(x_l-1)}{\theta(\theta+\bar a)}\right)^{-\theta}\right).$$

Using Proposition 3.2 and the relation $2\bar\sigma = \bar a + \bar b$ one transforms this into the desired result. □

Corollary 3.4. *In the hypotheses of Theorem 3.3, assume additionally that $a_n \ge b_n \ge -\frac12$. Then for any $k = 1, 2, \dots,$ the functions* (3.2) *converge to the function* (3.3) *uniformly on the torus $\mathbb{T}^k$.*

Proof. The assumption $a_n \ge b_n \ge -\frac12$ makes it possible to apply Proposition 1.1 which implies that the functions (2.6) are positive definite functions on the torus $\mathbb{T}^k$. Since the function (3.3) is real–analytic, our claim follows from Theorem 3.3 by virtue of a well–known general fact (see, e.g., Lemma 4.2 from [OO4]). □

4. Necessary conditions of regularity

The argument of this section is similar to that of [OO4, §5]. We begin with three technical lemmas which then are used to prove Proposition 4.4.

Lemma 4.1. *Assume $h \ge \theta n - 1/2$. Then we have*

$$I_{(2)}(\lambda_1, \dots, \lambda_n; \theta; h) \le \left(I_{(1)}(\lambda_1, \dots, \lambda_n; \theta; h)\right)^2$$

for any partition λ, $n = 1, 2, \dots,$ and $\theta \ge 0$.

Observe that if $h = \theta n + (a+b+1)/2$, where $a, b > -1$, then the hypothesis of the lemma is satisfied.

Proof. Set $A = I_{(1)}(\lambda; \theta; h)$ and $B = I_{(2)}(\lambda; \theta; h)$. From the combinatorial formula (2.4) we obtain

$$A = \sum_i \left((l_i + h - \theta i)^2 - (h - \theta i)^2\right),$$

$$B = \sum_i \left((l_i + h - \theta i)^2 - (h-\theta i)^2\right)\left((\lambda_i + h - \theta i)^2 - (h - \theta i + 1)^2\right) + \frac{2\theta}{1+\theta}\sum_{i<j}\left((l_j + h - \theta j)^2 - (h - \theta j)^2\right)\left((\lambda_i + h - \theta i)^2 - (h - \theta i + 1)^2\right).$$

It is elementary to check that for any $l \in \mathbb{Z}_{\geq 0}$ and any $i \leq n$ one has

$$(l+h-\theta i)^2 - (h-\theta i)^2 \geq 0$$

provided $h \geq \theta n - 1/2$; in particular,

$$(1+h-\theta i)^2 - (h-\theta i)^2 \geq 0\,.$$

Therefore

$$\begin{aligned} B \leq & \sum_i \left((\lambda_i+h-\theta i)^2-(h-\theta i)^2\right)\left((\lambda_i+h-\theta i)^2-(h-\theta i)^2\right) \\ &+ \frac{2\theta}{1+\theta}\sum_{i<j}\left((\lambda_j+h-\theta j)^2-(h-\theta j)^2\right)\left((\lambda_i+h-\theta i)^2-(h-\theta i)^2\right) \end{aligned}$$

(here we use the inequality with general l to conclude that the first factors are nonnegative, while the inequality with $l=1$ is used to remove 1 in the second factors). Since $\frac{2\theta}{1+\theta} < 2$ for any $\theta > -1$, the lemma follows. $\square$

Lemma 4.2. *Suppose that*

$$I_{(1)}(\lambda(n);\theta;h_n) = O(n^2)\,, \quad n \to \infty\,,$$

and

$$h_n \sim h_0 n\,, \quad h_0 > \theta/2, \quad n \to \infty\,.$$

Then

$$|\lambda(n)| = O(n)\,, \quad n \to \infty\,.$$

Observe that if $h_n = \sigma_n + \theta n$, where $\sigma_n \sim \bar\sigma n$, $\bar\sigma \geq 0$, then the hypothesis of the lemma is satisfied.

Proof. We have

$$\begin{aligned} I_{(1)}(\lambda;\theta;h_n) &= \sum_{i=1}^n \lambda_i(\lambda_i + 2h_n - 2\theta i) \\ &= \sum_{i=1}^n \lambda_i^2 + \theta \sum_{1\leq i<j\leq n}(\lambda_i-\lambda_j) + (2h_n - \theta(n+1))\sum_{i=1}^n \lambda_i\,. \end{aligned}$$

Since all summands are positive for large n we conclude that

$$(2h_0-\theta)\,n\sum_{i=1}^n \lambda_i(n) = O(n^2)\,, \quad n\to\infty\,,$$

which implies $|\lambda(n)| = O(n)$. $\square$

Lemma 4.3. *Let $\bar a \ge 0$ and $\bar b \ge 0$ be fixed. Assume $(\dot\alpha, \dot\beta, \dot\gamma)$ and $(\ddot\alpha, \ddot\beta, \ddot\gamma)$ are two systems of VK parameters such that the corresponding functions* (1.11) *coincide,*

$$e^{\dot\gamma(x-1)} \prod_{i=1}^{\infty} \frac{1 + \dfrac{\dot\beta_i}{2}\left(\dfrac{2\theta + \bar a + \bar b - \theta\dot\beta_i}{\theta + \bar a}\right)(x-1)}{\left(1 - \dfrac{\dot\alpha_i}{2\theta}\left(\dfrac{2\theta + \bar a + \bar b + \dot\alpha_i}{\theta + \bar a}\right)(x-1)\right)^{\theta}}$$
$$= e^{\ddot\gamma(x-1)} \prod_{i=1}^{\infty} \frac{1 + \dfrac{\ddot\beta_i}{2}\left(\dfrac{2\theta + \bar a + \bar b - \theta\ddot\beta_i}{\theta + \bar a}\right)(x-1)}{\left(1 - \dfrac{\ddot\alpha_i}{2\theta}\left(\dfrac{2\theta + \bar a + \bar b + \ddot\alpha_i}{\theta + \bar a}\right)(x-1)\right)^{\theta}}$$

for any $x \in [-1, 1]$.

Then $(\dot\alpha, \dot\beta, \dot\gamma) = (\ddot\alpha, \ddot\beta, \ddot\gamma)$.

Proof. A similar claim was established in step 3 of the proof of Theorem 5.1 from [OO4]. We shall use a similar argument.

Write the above identity of functions as

$$e^{\dot\gamma y} \prod_{i=1}^{\infty} \frac{1 + \dot B_i y}{\left(1 - \dot A_i y\right)^{\theta}} = e^{\ddot\gamma y} \prod_{i=1}^{\infty} \frac{1 + \ddot B_i y}{\left(1 - \ddot A_i y\right)^{\theta}}, \qquad y \in [-2, 0], \tag{4.1}$$

and observe that

$$\dot A_1 \ge \dot A_2 \ge \dots \ge 0, \qquad \dot B_1 \ge \dot B_2 \ge \dots \ge 0, \qquad \sum_{i=1}^{\infty} (\dot A_i + \dot B_i) < \infty$$
$$\ddot A_1 \ge \ddot A_2 \ge \dots \ge 0, \qquad \ddot B_1 \ge \ddot B_2 \ge \dots \ge 0, \qquad \sum_{i=1}^{\infty} (\ddot A_i + \ddot B_i) < \infty$$

Since the correspondences

$$\dot\alpha_i \leftrightarrow \dot A_i, \qquad \dot\beta_i \leftrightarrow \dot B_i, \qquad \ddot\alpha_i \leftrightarrow \ddot A_i, \qquad \ddot\beta_i \leftrightarrow \ddot B_i$$

are one–to–one, it suffices to prove that (4.1) implies

$$\dot A_i = \ddot A_i, \qquad \dot B_i = \ddot B_i, \qquad \dot\gamma = \ddot\gamma.$$

We may extend (4.1) to an identity between two holomorphic functions in y. These functions are well defined at least in the left half–plane $\Re y < 0$. Actually, the left–hand side is holomorphic in the half–plane $\Re y < (\dot A_1)^{-1}$ and has a singularity at $y = (\dot A_1)^{-1}$. Likewise, the right–hand side is holomorphic in the half–plane $\Re y < (\ddot A_1)^{-1}$ and has a singularity at $y = (\ddot A_1)^{-1}$. This implies $\dot A_1 = \ddot A_1$. Thus,

both sides of (4.1) have common factors which can be cancelled. Iterating this procedure we prove that $\dot{A}_2 = \ddot{A}_2$, etc. Then we come to an identity of entire functions,

$$e^{\dot{\gamma} y} \prod_{i=1}^{\infty} (1 + \dot{B}_i y) = e^{\ddot{\gamma} y} \prod_{i=1}^{\infty} (1 + \ddot{B}_i y).$$

Examining the zeros of both sides we see that $\dot{B}_i = \ddot{B}_i$ for all $i = 1, 2, \dots$, and finally we conclude that $\dot{\gamma} = \ddot{\gamma}$. $\square$

Theorem 4.4. *Let $a_n \geq b_n \geq -\frac{1}{2}$ and let $\{\lambda(n)\}$ be a sequence of partitions with $\ell(\lambda(n)) \leq n$. Assume that either $\{\lambda(n)\}$ is minimally regular, or the functions*

$$\Phi_{\lambda(n)}(z, \underbrace{1, \dots, 1}_{n-1 \text{ times}}; \theta, a_n, b_n), \quad |z| = 1, \tag{4.2}$$

converge infinitesimally about $z = 1 \in \mathbb{T}^1$, see Definition 1.2.

Then $\{\lambda(n)\}$ is a Vershik-Kerov sequence.

Proof. The proof given below is completely parallel to the proof of Theorem 5.1 in [OO4].

Step 1. Consider the following function on the unit circle $\mathbb{T}$

$$\phi_n(z) = \Phi_{\lambda(n)}(z, \underbrace{1, \dots, 1}_{n-1}; \theta, a_n, b_n), \qquad z \in \mathbb{T}.$$

By the binomial formula (2.7) and Proposition 2.4 we have

$$\phi_n(z) = 1 + A_{1,n}(z + z^{-1} - 2) + A_{2,n}(z + z^{-1} - 2)^2 + \dots,$$

where

$$A_{1,n} \sim \text{const}_1 \, \frac{I_{(1)}(\lambda(n); \theta n + \sigma_n)}{n^2}, \qquad A_{2,n} \sim \text{const}_2 \, \frac{I_{(2)}(\lambda(n); \theta n + \sigma_n)}{n^4}, \tag{4.3}$$

with some constants not depending on n.

We claim that

$$I_{(1)}(\lambda(n); \theta n + \sigma_n) = O(n^2). \tag{4.4}$$

Indeed, in case when the functions (4.2) converge infinitesimally, this bound is immediate. Let us prove it when $\{\lambda(n)\}$ is minimally regular. If $z = e^{i\varphi}$ then

$$z + z^{-1} - 2 = 2(\cos\varphi - 1) = -\varphi^2 + \tfrac{1}{12}\varphi^4 + O(\varphi^6)$$

so that

$$\begin{aligned} \phi_n(e^{i\varphi}) &= 1 + A_{1,n}(-\varphi^2 + \tfrac{1}{12}\varphi^4) + A_{2,n}\varphi^4 + O(\varphi^6) \\ &= 1 - A_{1,n}\varphi^2 + (A_{2,n} + \tfrac{1}{12}A_{1,n})\varphi^4 + O(\varphi^6). \end{aligned}$$

On the other hand, we know that $\phi_n(z)$ is a normalized positive definite function on $\mathbb{T}$ (see the proof of Corollary 3.4), hence it is the Fourier transform of a probability measure M_n on the lattice $\mathbb{Z}$. The assumption of minimal regularity means that the measures M_n weakly converge to a probability measure on $\mathbb{Z}$. In such a situation, Lemma 5.2 from [OO4] says that if the second moments of the measures M_n are not uniformly bounded then, passing to a suitable subsequence where the second moments tend to infinity, we obtain that the fourth moments grow faster than the squared second moments. Up to constant factors, the second and fourth moments are the coefficients in φ^2 and φ^4, respectively. This means that if the number sequence $\{A_{1,n}\}$ is unbounded then, for a suitable subsequence of indices n, $A_{2,n}$ grows faster than $(A_{1,n})^2$. But, by virtue of (4.3), this contradicts Lemma 4.1. We conclude that $\{A_{1,n}\}$ is bounded, which is equivalent to (4.4).

Step 2. We claim that

$$|\lambda(n)| = O(n)\,, \quad n \to \infty\,. \tag{4.5}$$

Indeed, this follows from the result of step 1 and Lemma 4.2.

Step 3. Using the bound (4.5) and Cantor's diagonal process, we see that any subsequence of $\{\lambda(n)\}$ contains a VK subsequence. It remains to prove that any two VK subsequences of $\{\lambda(n)\}$ have the same VK parameters. By Theorem 3.3 and the hypotheses of Theorem 4.4, both subsequences lead to one and the same limit function. Then we apply Lemma 4.3 to conclude that the VK parameters are the same. $\square$

Proof of Theorem 1.4. Let, as usual, $\{\lambda(n)\}$ be a sequence of partitions with $\ell(\lambda(n)) \le n$, and let the parameters a_n, b_n satisfy the inequality $a_n \ge b_n \ge -\frac{1}{2}$ and condition (1.7).

By Theorem 4.4, each of the 3 regularity properties of Definition 1.2 implies the VK property of Definition 1.3. Conversely, assume $\{\lambda(n)\}$ is a VK sequence. Then, by virtue of Theorem 3.3 and Corollary 3.4, this implies all 3 regularity properties of Definition 1.2. This proves claim (i) of Theorem 1.4. Corollary 3.4 also proves claim (ii) of Theorem 1.4. $\square$

5. The convex set Υ^θ

In this section we fix parameters a, b such that $a \ge b \ge -\frac{1}{2}$.

For $n = 1, 2, \dots$ let $\Upsilon_n^{\theta,a,b}$ be the set of functions on the torus $\mathbb{T}^n$ of the form

$$\varphi(z_1, \dots, z_n) = \sum_{\lambda\colon \ell(\lambda) \le n} c_\lambda \Phi_\lambda(z_1, \dots, z_n; \theta, a, b),$$

where

$$c_\lambda \ge 0, \qquad \sum_{\lambda\colon \ell(\lambda) \le n} c_\lambda = 1.$$

Note that the coefficients c_λ are uniquely determined by the function φ, because the Jacobi polynomials form an orthogonal basis in a suitable L^2 space. Recall that each $\Phi_\lambda(\,\cdot\,;\theta,a,b)$ is a positive definite function on the torus (see the proof of Corollary 3.4), normalized at the unit element $(1,\dots,1)\in\mathbb{T}^n$. It follows that $|\Phi_\lambda(z_1,\dots,z_n;\theta,a,b)|\le 1$, which implies that the series converges uniformly and defines a continuous function on $\mathbb{T}^n$. Thus, $\Upsilon_n^{\theta,a,b}$ is a subset of the set of continuous, positive definite, normalized functions on $\mathbb{T}^n$. It is clear that $\Upsilon_n^{\theta,a,b}$ is a convex set. As an abstract convex set, it is isomorphic to a simplex with infinitely many vertices.

Proposition 5.1. *Let λ be an arbitrary partition with with $\ell(\lambda)\le n$. Expand the function $\mathfrak{J}_\lambda(z_1,\dots,z_{n-1},1;\theta,a,b)$ in Jacobi polynomials in $n-1$ variables $z_1,\dots,z_{n-1}$, with the same parameters θ,a,b. Then all coefficients in this expansion are non-negative.*

Proof. This fact can be derived by a degeneration from the branching rule for Koornwinder polynomials, established by Rains [R, (5.76)]. □

By Proposition 5.1, the specialization $z_n=1$ sends $\Phi_\lambda(z_1,\dots,z_n;\theta,a,b)$ to a function from $\Upsilon_{n-1}^{\theta,a,b}$. Hence this specialization map determines an affine map $\Upsilon_n^{\theta,a,b}\to\Upsilon_{n-1}^{\theta,a,b}$. Using these maps for $n=2,3,\dots$ we set

$$\Upsilon^\theta=\varprojlim\Upsilon_n^{\theta,a,b},\qquad n\to\infty.$$

As we shall see (Corollary 5.3), this projective limit space does not depend on a,b.

An equivalent definition is as follows. Consider the set $\mathbb{T}_0^\infty=\varinjlim\mathbb{T}^n$ whose elements are infinite vectors $(z_1,z_2,\dots)\in\mathbb{T}\times\mathbb{T}\times\dots$ with finitely many coordinates z_i distinct from 1. Then elements of Υ^θ can be described as functions $\varphi(z_1,z_2,\dots)$ on $\mathbb{T}_0^\infty$ such that for any $n=1,2,\dots$, the function

$$\varphi_n(z_1,\dots,z_n)=\varphi(z_1,\dots,z_n,1,1,\dots)$$

on $\mathbb{T}^n$ belongs to $\Upsilon_n^{\theta,a,b}$.

It is clear that Υ^θ is a convex set. Let $\operatorname{Ex}\Upsilon^\theta$ denote the set of its extreme points.

Theorem 5.2. *There is a one–to–one correspondence between elements of the set $\operatorname{Ex}\Upsilon^\theta$ and collections (α,β,γ) of VK parameters,*

$$\alpha=(\alpha_1\ge\alpha_2\ge\dots\ge0),\quad \beta=(\beta_1\ge\beta_2\ge\dots\ge0),\quad \gamma\ge0,\quad \sum(\alpha_i+\beta_i)<\infty.$$

Given (α,β,γ), the corresponding element of $\operatorname{Ex}\Upsilon^\theta$, viewed as a function on $\mathbb{T}_0^\infty$, has the form

$$\Phi_{\alpha,\beta,\gamma}(z_1,z_2,\dots)=\prod_{j=1}^\infty\phi_{\alpha,\beta,\gamma}\left(\frac{z_j+z_j^{-1}}{2}\right)$$

where $(z_1, z_2, \dots) \in \mathbb{T}_0^\infty$, *and the function* $\phi_{\alpha,\beta,\gamma}$ *is as in Theorem 1.4,*

$$\phi_{\alpha,\beta,\gamma}\left(\frac{z+z^{-1}}{2}\right) = e^{\frac{\gamma}{2}(z+z^{-1}-2)} \prod_{i=1}^{\infty} \frac{\left(1+\frac{1}{2}\beta_i(z-1)\right)\left(1+\frac{1}{2}\beta_i(z^{-1}-1)\right)}{\left(\left(1-\frac{1}{2}\alpha_i(z-1)/\theta\right)\left(1-\frac{1}{2}\alpha_i(z^{-1}-1)/\theta\right)\right)^{\theta}}$$

or equivalently

$$\phi_{\alpha,\beta,\gamma}(x) = e^{\gamma(x-1)} \prod_{i=1}^{\infty} \frac{\left(1+\beta_i\left(1-\frac{\beta_i}{2}\right)\right)(x-1)}{\left\{\left(1-\frac{\alpha_i}{\theta}\left(1+\frac{\alpha_i}{2\theta}\right)\right)(x-1)\right\}^{\theta}}, \qquad x \in [-1,1].$$

In particular, $\operatorname{Ex}\Upsilon^\theta$ *does not depend on the parameters* a, b.

Proof. This can be proved exactly as Theorem 1.4 in [OO4], see §6 in [OO4]. By virtue of a general result (see Theorem 6.1 in [OO4]), each function $\Phi \in \operatorname{Ex}\Upsilon^\theta$ can be approximated (uniformly on any finite–dimensional sub–torus $\mathbb{T}^k$) by a sequence $\{\Phi_{\lambda(n)}(\,\cdot\,;\theta,a,b)\}$. By Theorem 1.4, the limit functions

$$\Phi = \lim_{n\to\infty} \Phi_{\lambda(n)}(\,\cdot\,;\theta,a,b)$$

are precisely the functions of the form $\Phi_{\alpha,\beta,\gamma}$. This shows that $\operatorname{Ex}\Upsilon^\theta \subset \{\Phi_{\alpha,\beta,\gamma}\}$.

The inverse inclusion $\{\Phi_{\alpha,\beta,\gamma}\} \subset \operatorname{Ex}\Upsilon^\theta$ is obtained by a simple argument using de Finetti's theorem. Indeed, the fact that each function $\Phi_{\alpha,\beta,\gamma}$ is positive definite on $\mathbb{T}_0^\infty$ and has the multiplicative form $\phi(z_1)\phi(z_2)\dots$ implies that $\Phi_{\alpha,\beta,\gamma}$ is an extreme point in a larger convex set, namely the set of characteristic functions of symmetric probability measures on $\mathbb{Z}^\infty = \mathbb{Z} \times \mathbb{Z} \times \dots$. □

Proposition 5.3. *Any element of the convex set* Υ^θ *is represented by a probability measure on the set* $\operatorname{Ex}\Upsilon^\theta$ *of extreme points, and this representation is unique.*

Proof. This claim does not follow directly from Choquet's theorem because the set Υ^θ is not compact. However, it can be checked, for instance, by the method of [O4, §9]. □

Corollary 5.4. *The set* Υ^θ *does not depend on the parameters* a, b.

Proof. Indeed, according to Theorem 5.2, $\operatorname{Ex}\Upsilon^\theta$ does not depend on a, b. Then the corollary follows from Proposition 5.3. □

6. Spherical functions on infinite–dimensional symmetric spaces

By an infinite–dimensional symmetric space we mean a homogeneous space G/K where the G and K are inductive limits of groups,

$$G = \varinjlim G(n), \quad K = \varinjlim K(n), \qquad n \to \infty,$$

such that for any n, $G(n)/K(n)$ is a Riemannian symmetric space of compact type belonging to one of the classical series. We assume that $\operatorname{rank}(G(n)/K(n)) = n$.

There are 10 such spaces G/K corresponding to 10 classical series of symmetric spaces, see Olshanski [O1], [O2]:

Table I

1. $G(n) = U(n)$, $K(n) = O(n)$.
2. $G(n) = U(n) \times U(n)$, $K(n) = U(n)$.
3. $G(n) = U(2n)$, $K(n) = Sp(n)$.
4. $G(n) = O(2n)$, $K(n) = O(n) \times O(n)$.
5. $G(n) = Sp(n)$, $K(n) = U(n)$.
6. $G(n) = U(2n)$, $K(n) = U(n) \times U(n)$.
7. $G(n) = O(\widetilde{n}) \times O(\widetilde{n})$, $K(n) = O(\widetilde{n})$.
8. $G(n) = Sp(n) \times Sp(n)$, $K(n) = Sp(n)$.
9. $G(n) = Sp(2n)$, $K = Sp(n) \times Sp(n)$.
10. $G(n) = O(2\widetilde{n})$, $K(n) = U(\widetilde{n})$.

Comments. a) The embeddings $G(n) \to G(n+1)$ and $K(n) \to K(n+1)$ which are implicit in the definition of the groups G and K are natural ones. The embeddings $K(n) \to G(n)$ are also quite evident. In particular, for series $2, 7, 8$, these are the diagonal embeddings.

b) For series 7 and 10, we wrote $\widetilde{n}$ instead of n because in these cases the rank equals $[\widetilde{n}/2]$. Here one may choose one of the two possible variants: $\widetilde{n} = 2n$ or $\widetilde{n} = 2n + 1$, and the inductive limit space G/K does not depend of the choice, up to isomorphism.

c) For the Grassmann spaces (series $4, 6, 9$), we could equally well use two distinct indices n_1, n_2. That is, we could deal with the spaces $O(n_1 + n_2)/O(n_1) \times O(n_2)$, $U(n_1 + n_2)/U(n_1) \times U(n_2)$, and $Sp(n_1 + n_2)/Sp(n_1) \times Sp(n_2)$. Again, such a generalization does affect the limit space G/K, provided that both indices go to infinity.

d) In the case of series $2, 7, 8$, the symmetric space $G(n)/K(n)$ is one of the classical groups $U(n)$, $O(n)$, $Sp(n)$, and the corresponding infinite–dimensional space G/K coincides with one of groups

$$U(\infty) = \varinjlim U(n), \qquad O(\infty) = \varinjlim O(n), \qquad Sp(\infty) = \varinjlim Sp(n).$$

As shown in [O1], [O2], for the 10 pairs (G, K) listed in Table I there is a rich theory of unitary representations. The present paper concerns a part of this theory related to spherical representations.

Assume (G, K) is one of the 10 pairs from Table I. Let T be a unitary representation of G in a Hilbert space H, and $\xi \in H$ be a distinguished unit K–invariant

vector. We say that (T, ξ) is a *spherical representation* of the pair (G, K) if ξ is a cyclic vector. That is, if the linear span of the vectors $T(g)\xi$, where g ranges over G, is dense in H. Two spherical representations, (T_1, ξ_1) and (T_2, ξ_2), are said to be *equivalent* if there is an isometry $H_1 \to H_2$ of the corresponding Hilbert spaces taking ξ_1 to ξ_2 and commuting with the action of G.

Attached to any spherical representation (T, ξ) is its *spherical function*

$$F(g) = (T(g)\xi, \xi), \qquad g \in G.$$

This is a positive definite function on the group G, two–sided invariant with respect to the subgroup K, and taking value 1 at the unity $e \in G$. Denote by $\Upsilon(G, K)$ the set of all functions with these three properties. Then the correspondence $T \to F$ defined above determines a *bijection* between equivalence classes of spherical representations and functions from the set $\Upsilon(G, K)$.

Of special interest are *irreducible* spherical representations. That is, those (T, ξ) for which T is an irreducible unitary representation of G. If T is an irreducible unitary representation of G then a K–invariant vector ξ (provided it exists) is *unique*, within a scalar factor which does not affect the spherical function. The spherical functions of irreducible spherical representations are precisely the *extreme points* of $\Upsilon(G, K)$ (it is worth noting that $\Upsilon(G, K)$ is a convex set).

Thus, classifying the irreducible spherical representations of (G, K) is equivalent to describing the extreme points of the convex set $\Upsilon(G, K)$. We aim to explain how this problem is related to that discussed in §6.

Let $\Upsilon(G(n), K(n))$ denote the set of functions on $G(n)$ that are positive definite, two–sided $K(n)$–invariant and take value 1 at the unity. This is a convex set, isomorphic to an infinite–dimensional simplex. Note that the vertices of this simplex, which are the extreme points of $\Upsilon(G(n), K(n))$, are the *indecomposable spherical functions* of $(G(n), K(n))$, that is, matrix coefficients of the form

$$(T^{(n)}(g)\xi^{(n)}, \xi^{(n)}), \qquad g \in G(n),$$

where $T^{(n)}$ is an arbitrary irreducible finite–dimensional unitary representation of $G(n)$ possessing a unit $K(n)$–invariant vector $\xi^{(n)}$ (such a vector is unique, within a scalar factor of absolute value 1).

The natural embedding of pairs

$$(G(n-1), K(n-1)) \hookrightarrow (G(n), K(n))$$

induces an affine projection

$$\Upsilon(G(n), K(n)) \;\to\; \Upsilon(G(n-1), K(n-1))$$

and we have

$$\Upsilon(G,K) = \varprojlim \Upsilon(G(n),K(n)), \qquad n\to\infty.$$

Below we focus on the series 4–10, the case of series 1–3 being the subject of our previous paper [OO4].

So, let $(G(n),K(n))$ belong to one of the seven series 4–10 from Table I. Let $\mathcal{R}_n$ denote the restricted root system of the symmetric space $(G(n),K(n))$. Then $\mathcal{R}_n$ coincides with one of the classical root system B_n, C_n, D_n or BC_n equipped with appropriate root multiplicities.

In all cases, it is convenient to regard $\mathcal{R}_n$ as a subsystem of the BC_n root system

$$R_n = \{\pm\varepsilon_i \pm \varepsilon_j : 1\le i<j\le n\} \cup \{\pm\varepsilon_i, \pm 2\varepsilon_i : 1\le i\le n\}\,.$$

To each of the series 4–10 one can attach a particular triple of parameters θ, a, b in such a way that the formal multiplicities k_α defined in (1.5) coincide with the true root multiplicities in $\mathcal{R}_n$. In particular, $k_\alpha = 0$ means that the root $\alpha\in R_n$ is not contained in $\mathcal{R}_n$.

Specifically, we have:

Table II

4. $G(n)=O(2n)$, $K(n)=O(n)\times O(n)$: $\theta=\frac12$, $a=b=-\frac12$, $\mathcal{R}_n=D_n$.
5. $G(n)=Sp(n)$, $K(n)=U(n)$: $\theta=\frac12$, $a=b=0$, $\mathcal{R}_n=C_n$.
6. $G(n)=U(2n)$, $K(n)=U(n)\times U(n)$: $\theta=1$, $a=b=0$, $\mathcal{R}_n=C_n$.
7. $G(n)=O(\widetilde n)\times O(\widetilde n)$, $K(n)=O(\widetilde n)$: $\theta=1$, $a=-\frac12$ or $\frac12$, $b=-\frac12$, $\mathcal{R}_n=D_n$ or B_n.
8. $G(n)=Sp(n)\times Sp(n)$, $K(n)=Sp(n)$: $\theta=1$, $a=b=\frac12$, $\mathcal{R}_n=C_n$.
9. $G(n)=Sp(2n)$, $K=Sp(n)\times Sp(n)$: $\theta=2$, $a=b=1$, $\mathcal{R}_n=C_n$.
10. $G(n)=O(2\widetilde n)$, $K(n)=U(\widetilde n)$: $\theta=2$, $a=0$ or 2, $b=0$, $\mathcal{R}_n=C_n$ or BC_n.

Here, in the case of series 7 and 10, the first option for a and $\mathcal{R}_n$ is chosen if $\widetilde n=2n$, and the second option is chosen if $\widetilde n = 2n+1$.

Proposition 6.1. *Let $(G(n),K(n))$ belong to one of the series 4–10 and let θ,a,b be the corresponding parameters as listed above. Assume additionally that the restricted root system $\mathcal{R}_n$ is not D_n. Then there are natural bijections*

$$\Upsilon_n^{\theta,a,b} \longleftrightarrow \Upsilon(G(n),K(n)), \qquad n=1,2,\dots \tag{6.1}$$

which are isomorphisms of convex sets and commute with the projections

$$\Upsilon_n^{\theta,a,b}\to\Upsilon_{n-1}^{\theta,a,b}, \qquad \Upsilon(G(n),K(n))\to\Upsilon(G(n-1),K(n-1)). \tag{6.2}$$

Idea of proof. Let $\mathcal{W}_n$ denote the restricted Weyl group of the symmetric space $G(n)/K(n)$. For any classical series, $\mathcal{W}_n$ may be identified with the BC_n Weyl

group $S(n) \ltimes \mathbb{Z}_n^2$ (realized as the group W_* acting on the torus $\mathbb{T}^n$ as explained in §1) or with its subgroup of index 2 (the latter possibility holds exactly when $\mathcal{R}_n = D_n$).

It is well known that the indecomposable spherical functions of $(G(n), K(n))$ can be interpreted as normalized $\mathcal{W}_n$–invariant orthogonal polynomials on the torus $\mathbb{T}^n$ with the weight (1.1), where the parameters θ, a, b are those attached to the corresponding series. When $\mathcal{R}_n \neq D_n$, these are precisely the normalized polynomials $\Phi_\lambda(z_1, \dots, z_n; \theta, a, b)$.

This yields the required bijection (6.1). The fact that these bijections are compatible with the projections (6.2) is readily verified. □

Corollary 6.2. *Under the hypotheses of Proposition 6.1 we have: the convex set $\Upsilon(G, K)$ is isomorphic to the convex set Υ^θ described in §5. In particular, it depends only on θ but not on a, b.*

Proof. Indeed, this follows at once from Proposition 6.1 and the results of §5. □

Remark 6.3. When the restricted root system $\mathcal{R}_n$ is of type D_n, certain polynomials $\Phi_\lambda(z_1, \dots, z_n; \theta, a, b)$ turn out to be half–sums of two distinct $\mathcal{W}_n$–invariant orthogonal polynomials. As a consequence, the set $\Upsilon(G(n), K(n))$ turns out to be somewhat larger than the set $\Upsilon_n^{\theta,a,b}$. Nevertheless, the claim of Corollary 6.2 holds in this case as well. Indeed, as is seen from the list above, the equality $\mathcal{R}_n = D_n$ occurs in two cases: for series 4 and for series 7 with $\widetilde{n}$ even. In the latter case we may choose $\widetilde{n}$ odd without changing the limit space G/K. In the former case the same effect is achieved if we take $G(n) = O(2n + m)$, $K(n) = O(n + m) \times O(n)$, where m is an arbitrary fixed positive integer.

7. The BC_n polynomials with $\theta = 1$

When $\theta = 1$, the multivariate Jacobi polynomials $\mathfrak{J}_\lambda$ and the interpolation polynomials I_μ admit explicit determinantal expressions. This makes it possible to establish the basic facts about these polynomials independently of the general theory, in a rather elementary way.

Throughout the present section we assume $\theta = 1$ and fix arbitrary parameters $a > -1, b > -1$. We give explicit formulas for both kinds of polynomials and sketch elementary proofs of the binomial formula (2.7) and of the branching rule for Jacobi polynomials, which in turns implies Proposition 5.1.

Let $\mathfrak{p}_l(x; a, b)$ (where $l = 0, 1, 2, \dots$) denote the classical Jacobi polynomials in a single variable x, orthogonal on the segment $-1 \le x \le 1$ with the weight function $(1 - x)^a(1 + x)^b$. We use the same normalization as in Erdelyi et al. [Er]. An explicit expression for $\mathfrak{p}_l(x; a, b)$ in terms of the Gauss hypergeometric function ${}_2F_1$

is

$$\mathfrak{p}_l(x;a,b)=\frac{\Gamma(l+a+1))}{\Gamma(l+1)\Gamma(a+1)}\,{}_2F_1(-l,l+a+b+1;a+1;\tfrac{1-x}{2}). \tag{7.1}$$

In particular, the value at $x=1$ is given by

$$\mathfrak{p}_l(1;a,b)=\frac{\Gamma(l+a+1))}{\Gamma(l+1)\Gamma(a+1)}$$

and the leading coefficient in $\mathfrak{p}_l(x;a,b)$ is

$$\varkappa(l;a,b)=2^{-l}\,\frac{\Gamma(2l+a+b+1)}{\Gamma(l+a+b+1)\Gamma(l+1)}\,.$$

More generally, for any $n=1,2,\dots$ and any partition λ with $\ell(\lambda)\le n$ we set

$$\mathfrak{P}_\lambda(x_1,\dots,x_n;a,b)=\frac{\det\limits_{1\le i,j\le n}\left[\mathfrak{p}_{\lambda_i+n-i}(x_j\,;a,b)\right]}{V(x)}\,, \tag{7.2}$$

where

$$V(x)=\prod_{1\le i<j\le n}(x_i-x_j).$$

Proposition 7.1. *The BC_n orthogonal polynomials with $\theta=1$ and arbitrary parameters $a,b>-1$ are expressed through the polynomials* (7.2) *as follows*

$$\mathfrak{J}_\lambda(z_1,\dots,z_n;1,a,b)=\text{const}\ \mathfrak{P}_\lambda\left(\frac{z_1+z_1^{-1}}{2},\dots,\frac{z_n+z_n^{-1}}{2}\,;a,b\right)$$

where

$$\text{const}=\frac{2^{|\lambda|}}{\prod_{i=1}^n\varkappa(\lambda_i+n-i;a,b)}$$

This fact is undoubtedly well known. E.g., in an equivalent form, it was pointed out in Lassalle [L]. For reader's convenience we present a proof.

Proof. It is readily verified that the polynomials (7.2) are pairwise orthogonal on the n–dimensional cube $[-1,1]^n$ with respect to the measure

$$V^2(x)\prod_{1\le i\le n}(1-x_i)^a(1+x_i)^b\,dx_1\dots,dx_n$$

which implies that the polynomials

$$(z_1,\dots,z_n)\mapsto\mathfrak{P}_\lambda\left(\frac{z_1+z_1^{-1}}{2},\dots,\frac{z_n+z_n^{-1}}{2}\,;a,b\right)$$

are pairwise orthogonal on the torus $\mathbb{T}^n$ with the weight (1.1) specialized at $\theta=1$.

Next, we have

$$\mathfrak{P}_\lambda(x_1,\dots,x_n;a,b)=\text{const}'\ x_1^{\lambda_1}\dots x_n^{\lambda_n}+\dots$$

where dots mean lower terms in lexicographic order and

$$\text{const}' = \prod_{i=1}^{n} \varkappa(\lambda_i + n - i; a, b),$$

which implies the triangularity condition (1.2) for the polynomials on the torus, defined by the right–hand side of (7.2).

The stronger triangularity condition (1.4) can also be readily verified. Indeed, comparing the determinantal expression (7.2) with the determinantal formula for the Schur polynomials

$$s_\mu(x_1, \dots, x_n) = \frac{\det\limits_{1 \le i,j \le n} \left[x_j^{\mu_i + n - i} \right]}{V(x)},$$

we see that

$$\mathfrak{P}_\lambda(x_1, \dots, x_n; a, b) = \sum_{\mu \subseteq \lambda} a_{\lambda\mu} s_\mu(x_1, \dots, x_n)$$

where $\mu \subseteq \lambda$ means that the diagram of μ is contained in that of λ, and $a_{\lambda\mu}$ are certain coefficients. Next, recall the well–known triangularity property of the Schur polynomials:

$$s_\mu(x_1, \dots, x_n) = \sum_{\nu \le \mu} K_{\mu\nu} m_\nu(x_1, \dots, x_n),$$

where m_ν is the conventional monomial symmetric function, $K_{\mu\nu}$ are certain coefficients (the Kostka numbers) and $\nu \le \mu$ is the dominance order on partitions ($\mu - \nu$ can be written as a linear combination of the vectors $\varepsilon_i - \varepsilon_j$, $i < j$).

From the last two formulas we obtain the triangularity condition of the form (1.4):

$$\begin{aligned} \mathfrak{P}_\lambda \left(\frac{z_1 + z_1^{-1}}{2}, \dots, \frac{z_n + z_n^{-1}}{2}; a, b \right) &= \sum_{\nu \ll \lambda} b_{\lambda\nu} m_\nu \left(\frac{z_1 + z_1^{-1}}{2}, \dots, \frac{z_n + z_n^{-1}}{2} \right) \\ &= \sum_{\mu \ll \lambda} c_{\lambda\mu} \widetilde{m}_\nu(z_1, \dots, z_n) \end{aligned}$$

with certain coefficients $b_{\lambda\nu}$ and $c_{\lambda\mu}$.

Thus, the polynomials in $(z_1, \dots, z_n)$ defined by the right–hand of (7.2) possess the characteristic properties of the BC_n orthogonal polynomials on the torus $\mathbb{T}^n$ with parameter $\theta = 1$ and hence coincide with the polynomials $\mathfrak{J}_\lambda(z_1, \dots, z_n; 1, a, b)$. □

By virtue of (7.2), Proposition 7.1 provides an explicit determinantal expression for the polynomials $\mathfrak{J}_\lambda(z_1, \dots, z_n; 1, a, b)$. Now we shall give an explicit expression of the BC_n interpolation polynomials with $\theta = 1$.

We need a notation. Given an infinite sequence of parameters $A = (A_1, A_2, \dots)$, define "generalized powers" of a variable y by

$$(y \mid A)^m = (y - A_1) \dots (y - A_m), \quad m = 1, 2, \dots; \qquad (y \mid A)^0 = 1.$$

Next, we define (generalized) *factorial Schur polynomials* in n variables by

$$s_\mu(y_1, \dots, y_n \mid A) = \frac{\det\limits_{1 \le i,j \le n} \left[(y_i \mid A)^{\mu_j + n - j} \right]}{V(y_1, \dots, y_n)}, \tag{7.3}$$

where μ is an arbitrary partition with $\ell(\mu) \le n$. Note that $s_\mu(y_1, \dots, y_n \mid A)$ is an inhomogeneous symmetric polynomial whose top degree homogeneous component is the conventional Schur polynomial $s_\mu(y_1, \dots, y_n)$. The polynomials (7.3) share many properties of the conventional Schur polynomials. A number of formulas for the polynomials (7.3) can be found in Macdonald [Ma2] and [Ma3, Example I.3.20], Molev [Mo], and also, for the special case $A = (0, 1, 2, \dots)$, in Okounkov–Olshanski [OO1]. Note that our notation for the parameters $A_1, A_2, \dots$ differs from that of Macdonald by sign.

Proposition 7.2. *The BC_n interpolation polynomials with $\theta = 1$ are expressed through the factorial Schur polynomials* (7.3) *as follows*

$$I_\mu(x_1, \dots, x_n; 1; h) = s_\mu((x_1 + h - 1)^2, \dots, (x_n + h - n)^2 \mid A), \tag{7.1}$$

where

$$A = ((h - n)^2, (h - n + 1)^2, (h - n + 2)^2, \dots).$$

Proof. We have to check that in the special case $\theta = 1$, all claims of Proposition 2.1 hold and the corresponding polynomials coincide with those given by formula (7.3). This was shown, in an elementary way, in Okounkov–Olshanski [OO3, Theorem 2.5]. $\square$

About this result, see also [Ok2, §3.3].

Proposition 7.3. *The polynomials $I_\mu(x_1, \dots, x_n; 1; h)$ defined by* (7.1) *satisfy the combinatorial formula of Proposition 2.2. That is,*

$$I_\mu(x_1, \dots, x_n; 1; h) = \sum_T \prod_{(i,j) \in \mu} \Big[\big(x_{T(i,j)} + h - T(i,j)\big)^2 - \big(j - i + h - T(i,j)\big)^2 \Big],$$

summed over all reverse tableaux T of shape μ, with entries in $\{1, \dots, n\}$.

Proof. This is a special case of the combinatorial formula for factorial Schur polynomials,

$$s_\mu(y_1,\dots,y_n \mid A) = \sum_T \prod_{(i,j)\in\mu} (y_{T(i,j)} - A_{j-i+n+1-T(i,j)}),$$

summed over all reverse tableaux T of shape μ, with entries in $\{1,\dots,n\}$. About the latter formula, see Goulden–Greene [GG], Macdonald [Ma2], and also Okounkov [Ok2, §3.3]. □

The binomial formula (2.7) of Proposition 2.3 reduces to the following claim

Proposition 7.4. *Let $\ell(\lambda) \le n$. We have*

$$\frac{\mathfrak{P}_\lambda(x_1,\dots,x_n;a,b)}{\mathfrak{P}_\lambda(\underbrace{1,\dots,1}_n;a,b)} = \sum_\mu \frac{I_\mu(\lambda;1;\sigma+n)\, s_\mu(x_1-1,\dots,x_n-1)}{c(n,\mu;a)}\,.$$

where $I_\mu(\lambda;1;\sigma+n)$ is the result of specializing $h=\sigma+n$ in $I_\mu(\lambda;1;h)$, $\sigma = (a+b+1)/2$, and

$$c(n,\mu;a) = 2^{|\mu|}\prod_{i=1}^n \frac{\Gamma(\mu_i+n-i+1)\Gamma(\mu_i+n-i+a+1)}{\Gamma(n-i+1)\Gamma(n-i+a+1)}$$

Proof. By virtue of the definition of the polynomials $I_\mu(\,\cdot\,;1,h)$ this formula can be rewritten as

$$\frac{\mathfrak{P}_\lambda(x_1,\dots,x_n;a,b)}{\mathfrak{P}_\lambda(\underbrace{1,\dots,1}_n;a,b)} = \sum_\mu \frac{s_\mu(l_1^2,\dots,l_n^2 \mid \sigma^2,(\sigma+1)^2,(\sigma+2)^2,\dots)\, s_\mu(x_1-1,\dots,x_n-1)}{c(n,\mu;a)}$$

where

$$l_i = \lambda_i + n - i + \sigma, \qquad i=1,\dots,n.$$

The latter formula can be directly derived from the determinantal formula (7.2) and the expression (7.1) for the classical Jacobi polynomials. For some special values of the parameters a,b (which correspond to characters of classical groups of the B, C, D series) such a computation was done in Okounkov–Olshanski [OO3, Theorem 1.2]. For general a,b the argument is quite similar. □

The next result is a special case of Proposition 5.1.

Proposition 7.5. *Let $n = 2, 3, \dots$ and λ be a partition with $\ell(\lambda) \le n$. In the expansion*

$$\mathfrak{P}_\lambda(x_1, \dots, x_{n-1}, 1; a, b) = \sum_{\nu\colon\, \ell(\nu) \le n-1} (\dots)\, \mathfrak{P}_\nu(x_1, \dots, x_{n-1}; a, b)$$

all coefficients $(\dots)$ are nonnegative.

Proof. a) First, let us describe the scheme of the proof. We shall use certain renormalized polynomials

$$\begin{aligned} R_\lambda(x_1, \dots, x_n) &= (\text{a positive factor})\, \mathfrak{P}_\lambda(x_1, \dots, x_n; a, b) \\ \widetilde{R}_\mu(x_1, \dots, x_{n-1}) &= (\text{a positive factor})\, \mathfrak{P}_\lambda(x_1, \dots, x_{n-1}; a+1, b). \end{aligned} \tag{7.4}$$

Below μ and ν denote partitions with $\ell(\,\cdot\,) \le n-1$. We shall establish the following two–step branching rule, which implies the proposition:

$$R_\lambda(x_1, \dots, x_{n-1}, 1) = \sum_{\mu \prec \lambda} \widetilde{R}_\mu(x_1, \dots, x_{n-1}) \tag{7.5}$$

$$\widetilde{R}_\mu(x_1, \dots, x_{n-1}) = \sum_{\nu \prec \mu \cup 0} A(\mu, \nu)\, R_\nu(x_1, \dots, x_{n-1}), \qquad A(\mu, \nu) > 0, \tag{7.6}$$

where $\mu \prec \lambda$ means

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \dots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n$$

and $\nu \prec \mu \cup 0$ means

$$\mu_1 \ge \nu_1 \ge \mu_2 \ge \dots \ge \mu_{n-1} \ge \nu_{n-1} \ge 0.$$

The coefficients $A(\mu, \nu)$ are specified below, see (7.13) and (7.14).

The proof presented below is quite elementary (we found it in 1998). It is worth noting that a much more general two–step branching rule, in the context of Koornwinder polynomials, was established by Rains, see [R, (5.76)].

b) Now we proceed to the proof of the proposition. For $l = 0, 1, 2, \dots$ set

$$r_l(x) = \frac{\mathfrak{p}_l(x; a, b)}{\mathfrak{p}_l(1; a, b)}, \qquad \widetilde{r}_l(x) = \frac{r_{l+1}(x) - r_l(x)}{x - 1}\,. \tag{7.7}$$

We have

$$\begin{aligned} r_l(x) &= (\text{a positive factor})\, \mathfrak{p}_l(x; a, b), \\ \widetilde{r}_l(x) &= (\text{a positive factor})\, \mathfrak{p}_l(x; a+1, b). \end{aligned} \tag{7.8}$$

Indeed, the first relation in (7.8) is evident, because $\mathfrak{p}_l(1; a, b) > 0$. Let us check the second relation.

Since $r_{l+1}(1)=r_l(1)=1$, it is clear that $\widetilde{r}_l(x)$ is a polynomial, and its degree is strictly equal to l. We have

$$\begin{aligned}&\int_{-1}^{1} x^m \widetilde{r}_l(x)(1-x)^{a+1}(1+x)^b dx\\ &= -\int_{-1}^{1} x^m r_{l+1}(x)(1-x)^a(1+x)^b dx + \int_{-1}^{1} x^m r_l(x)(1-x)^a(1+x)^b dx,\end{aligned}$$

and the latter two integrals vanish whenever $m<n$. This implies that $\widetilde{r}_l(x)$ is proportional to $\mathfrak{p}_l(x;a+1,b)$.

By the definition of $\widetilde{r}_l(x)$, its leading coefficient is the same as that of $r_{l+1}(x)$, hence positive. This completes the proof of (7.8).

c) Set

$$R_\lambda(x_1,\dots,x_n)=\frac{\det\limits_{1\le i,j\le n}\left[r_{\lambda_i+n-i}(x_j)\right]}{V(x_1,\dots,x_n)} \tag{7.9}$$

$$\widetilde{R}_\mu(x_1,\dots,x_{n-1})=\frac{\det\limits_{1\le i,j\le n-1}\left[\widetilde{r}_{\mu_i+n-1-i}(x_j)\right]}{V(x_1,\dots,x_{n-1})} \tag{7.10}$$

Then, due to (7.8), we have (7.4).

d) Let us check the first step of our branching rule, (7.5). By virtue of (7.9), the left–hand side of (7.5) is given by the ratio of a determinant of the order n and a Vandermonde. Examine the determinant in the numerator. For $i=1,\dots,n-1$, let us subtract the $i+1$th row from the ith row. Then we come to a determinant of order $n-1$. Its (i,j)th entry is equal to

$$r_{\lambda_i+n-i}(x_j)-r_{\lambda_{i+1}+n-1-i}(x_j),$$

and we may divide it by (x_j-1), because of the obvious relation

$$V(x_1,\dots,x_{n-1},1)=V(x_1,\dots,x_{n-1})\prod_{j=1}^{n-1}(x_j-1).$$

Then the (i,j) entry will take the form

$$\frac{r_{l_1+1}(x_j)-r_{l_2}(x_j)}{x_j-1}=\sum_{m=l_2}^{l_1}\widetilde{r}_m(x_j),$$

where we abbreviated $l_1=\lambda_i+n-i-1$, $l_2=\lambda_{i+1}+n-1-i$.

Employing the latter expression and expanding the determinant along the rows we get the desired result (7.5).

e) Let us check the second step of the branching rule, (7.6).

Write the three–term recurrence relation for the polynomials r_m:

$$r_{m+1}(x) = (a_m x + b_m) r_m(x) - c_m r_{m-1}(x), \qquad m \geq 1. \tag{7.11}$$

By the normalization,

$$a_m + b_m - c_m = 1.$$

By making use of these relations and the definition of $\widetilde{r}_m$ we get

$$\widetilde{r}_m = a_m r_m + c_m \widetilde{r}_{m-1}, \qquad m \geq 1.$$

Iterating this relation we further get

$$\widetilde{r}_m = \sum_{l=k}^{m} B(m,l) r_l + c_m \dots c_k \widetilde{r}_{k-1}, \qquad m \geq k \geq 0, \tag{7.12}$$

where

$$B(m,l) = \left(\prod_{l<p\leq m} c_p \right) a_l\,, \qquad m \geq l \geq 0.$$

When $k=0$ we agree that $\widetilde{r}_{-1}=0$ so that the last term in (7.12) disappears.

By virtue of (7.9) and (7.10), the relation (7.6) is equivalent to

$$\det_{1\leq i,j\leq n-1} [\widetilde{r}_{\mu_i+n-1-i}(x_j)] = \sum_{\nu \prec \mu \cup 0} A(\mu,\nu) \det_{1\leq i,j\leq n-1} [r_{\nu_i+n-1-i}(x_j)]$$

Examine the determinant in the left–hand side. Its entries in the ith row are of the form $\widetilde{r}_{\mu_i+n-1-i}(x_j)$, where $j=1,\dots,n-1$. We shall apply to them the relation (7.12), taking $m=\mu_i+n-1-i$, $k=\mu_{i+1}+n-1-i$.

First, do this for the first row, $i=1$. We get a decomposition of the form

$$\widetilde{r}_{\mu_1+n-2}(x_j) = \sum_{l=\mu_2+n-2}^{\mu_1+n-2} (...) r_l(x_j) + (...) \widetilde{r}_{\mu_2+n-3}(x_j),$$

where the coefficients marked as (...) are expressed through the c– and a–coefficients. Remark that the last term coincides, within a scalar factor, with that in the second row. Consequently, when we expand the determinant along the first row, it will play no role.

Now, we perform this expansion and then look at the second row and repeat the same procedure, etc. For the ith row ($i=1,\dots,n-2$) we get the decomposition

$$\widetilde{r}_{\mu_i+n-1-i}(x_j) = \sum_{l=\mu_{i+1}+n-1-i}^{\mu_i+n-1-i} (...) r_l(x_j) + (...) \widetilde{r}_{\mu_{i+1}+n-2-i}(x_j),$$

and when we come to the last row ($i = n-1$) then we choose $k = 0$ so that the "last term" mentioned above will disappear at all. Finally we get the desired expression (7.6) with coefficients $A(\mu, \nu)$ given by

$$A(\mu,\nu) = \prod_{i=1}^{n-1} B(\mu_i + n - 1 - i,\, \nu_i + n - 1 - i)\,. \tag{7.13}$$

e) To check that these coefficients are strictly positive we use a well–known general property of orthogonal polynomials: the coefficients a_m, c_m in the three–term relation (7.11) are strictly positive provided that the leading coefficients of the polynomials are strictly positive (see [Er], section 10.3, formulas (7) and (8)). The latter property holds in our case, so that we conclude that $B(m, l) > 0$ for any $m \ge l \ge 0$, and finally $A(\mu, \nu) > 0$.

f) The coefficients $B(m, l)$ entering the formula (7.13) can be explicitly computed:

$$B(m,l) = \frac{(2m+a+b)\Gamma(m+b+1)m!(2l+a+b+1)\Gamma(l+a+b+1)\Gamma(l+a+1)}{2\Gamma(m+a+b+2)\Gamma(m+a+2)\Gamma(n+b+1)l!}\,. \tag{7.14}$$

When $l = 0$, the product $(2l + a + b + 1)\Gamma(l + a + b + 1)$ must be replaced by $\Gamma(l + a + b + 2) = \Gamma(a + b + 2)$.

Indeed, let k_m be the leading coefficients in $\mathfrak{p}_m(x; a, b)$ and h_m be the squared norm of $\mathfrak{p}_m(x; a, b)$ (this is the standard notation, see [Er, §10.3]), and set also $e_m = \mathfrak{p}_m(1; a, b)$. It follows from [Er, §10.3 (8)] and (7.7) that

$$a_m = \frac{k_{m+1}e_m}{k_m e_{m+1}}\,, \qquad c_m = \frac{k_{m+1}k_{m-1}h_m e_{m-1}}{k_m^2 h_{m-1} e_{m+1}}\,,$$

whence

$$B(m,l) = \frac{k_{m+1}h_m e_l^2}{k_m h_l e_m e_{m+1}}\,.$$

Using the explicit values of the constants entering this formula (see [Er, §10.8]) we get (7.14). □

References

[BeO] R. J. Beerends and E. M. Opdam, *Certain hypergeometric series related to the root system BC*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 581–609.

[BorO] A. Borodin and G. Olshanski, *Harmonic analysis on the infinite–dimensional unitary group and determinantal point processes*, Ann. Math. **161** (2005), no. 3, 1319–1422; `arXiv: math.RT/0109194`.

[Boy] R. Boyer, *Characters and factor representations of the infinite dimensional classical groups*, J. Operator Theory **28** (1992), 281–307.

[Di] J. F. van Diejen, *Commuting difference operators with polynomial eigenfunctions*, Compositio Mathematica **95** (1995), 183–233.

[Du] E. Dueñez, *Random matrix ensembles associated to compact symmetric spaces*, Comm. Math. Phys. **244** (2004), 29–61; `arXiv: math-ph/0111005`.

[Er] A. Erdelyi et al., *Higher transcendental functions, vol. 2*, Mc Graw–Hill Book Company, 1953.

[GG] I. Goulden and C. Greene, *A new tableau representation for supersymmetric Schur functions*, J. Algebra **170** (1994), 687–703.

[HS] G. Heckman and H. Schlichtkrull, *Harmonic analysis and special functions on symmetric spaces*, Academic Press, San Diego, 1994.

[JC] A. T. James and A. G. Constantine, *Generalized Jacobi polynomials as spherical functions of the Grassmann manifold*, Proc. London Math. Soc. **29** (1974), 174–192.

[KOO] S. Kerov, A. Okounkov, and G. Olshanski, *The boundary of Young graph with Jack edge multiplicities*, Intern. Math. Res. Notices (1998), no. 4, 173–199; `arXiv: q-alg/9703037`.

[K1] T. H. Koornwinder, *Askey–Wilson polynomials for root system of type BC*, in: Hypergeometric functions on domains of positivity, Jack polynomials, and applications, Contemp. Math., vol. 138, Amer. Math. Soc., 1992.

[K2] T. H. Koornwinder, *Special functions associated with root systems: A first introduction for non–specialists*, In: Special functions and differential equations (K. Srinivasa Rao et al., eds.), Allied Publishers, Madras, 1998, pp. 10–24; also available via Koornwinder's web page `www.sience.uva.nl/~thk/art/`.

[L] M. Lassalle, *Polynômes de Jacobi généralisés*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 6, 425–428.

[Ma1] I. G. Macdonald, *Hypergeometric functions*, unpublished manuscript, about 1987.

[Ma2] I. G. Macdonald, *A new class of symmetric functions*, Séminaire Lotharingien de Combinatoire **20** (1988), Article B20a, available via `www.emis.de/journals/SLC/`.

[Ma3] I. G. Macdonald, *Symmetric functions and Hall polynomials, second edition*, Oxford University Press, 1995.

[Ma4] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, Preprint, 1987; reproduced in: Séminaire Lotharingien de Combinatoire **45** (2000), Article B45a, available via `www.emis.de/journals/SLC/`.

[Mo] A. Molev, *Factorial supersymmetric Schur functions and super Capelli identities*, Kirillov's Seminar on Representation Theory (G. Olshanski, ed.), American Mathematical Society Translations (2), Vol. 181, Amer. Math. Soc., Providence, R.I., 1997, pp. 109–137.

[Ok1] A. Okounkov, *BC_n–type shifted Macdonald polynomials and binomial formula for Koornwinder polynomials*, Transformation Groups **3** (1998), 181–207; `arXiv: q-alg/9611011`.

[Ok2] A. Okounkov, *On Newton interpolation of symmetric functions: a characterization of interpolation Macdonald polynomials*, Adv. Appl. Math. **20** (1998), 395–428; `arXiv: q-alg/9712052`.

[OO1] A. Okounkov and G. Olshanski, *Shifted Schur functions*, Algebra and Analysis 9 (1997), no. 2, 73–146 (Russian); English translation: St. Petersburg Math. J., **9** (1998), 239–300; `arXiv: q-alg/9605042`.

[OO2] A. Okounkov and G. Olshanski, *Shifted Jack polynomials, binomial formula, and applications*, Math. Res. Lett. **4** (1997), 69–78; `arXiv: q-alg/9608020`.

[OO3] A. Okounkov and G. Olshanski, *Shifted Schur functions II*, In: Kirillov's Seminar on Representation Theory (G. Olshanski, ed.), American Mathematical Society Translations, Ser. 2, Amer. Math. Soc., Providence, R.I., 1998, pp. 245–271; `arXiv: q-alg/9612025` .

[OO4] A. Okounkov and G. Olshanski, *Asymptotics of Jack polynomials as the number of variables goes to infinity*, Intern. Math. Res. Notices (1998), no. 13, 641–682; `arXiv: q-alg/9709011`.

[OP] M. A. Olshanetsky and A. M. Perelomov, *Quantum systems related to root systems, and radial parts of Laplace operators*, Funct. Anal. Appl. **12** (1978), 121–128; `arXiv: math-ph/0203031`.

[O1] G. Olshanski, *Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe*, Soviet Math. Doklady **27** (1983), no. 2, 290–294.

[O2] G. Olshanski, *Unitary representations of infinite–dimensional pairs (G, K) and the formalism of R. Howe*, In: Representation of Lie Groups and Related Topics (A. Vershik and D. Zhelobenko, eds.), Advanced Studies in Contemporary Math. **7**, Gordon and Breach Science Publishers, New York etc., 1990, pp. 269–463.

[O3] G. Olshanski, *On semigroups related to infinite–dimensional groups*, In: Topics in representation theory (A. A. Kirillov, ed.). Advances in Soviet Math., vol. 2. Amer. Math. Soc., Providence, R.I., 1991, 67–101.

[O4] G. Olshanski, *The problem of harmonic analysis on the infinite–dimensional unitary group*, J. Funct. Anal. **205** (2003), no. 2, 464–524; `arXiv: math.RT/0109193`.

[P] D. Pickrell, *Separable representations for automorphism group of infinite symmetric spaces*, J. Func. Anal. **90** (1990), 1–26.

[R] E. M. Rains, BC_n*–symmetric polynomials*, Transformation Groups **10** (2005), no. 1, 63-132; `arXiv: math.QA/0112035`.

[St] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. in Math. **77** (1989), 76–115.

[VK] A. M. Vershik and S. V. Kerov, *Characters and factor representations of the infinite unitary group*, Soviet Math. Doklady **26** (1982), 570–574.

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